IMPORTANT INFORMATION ON MATH 603

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Contents

1. Introduction

Representation theory studies *representations* of groups, associative algebras, Lie algebras etc., i.e. homomorphisms from a given group/algebra to the "general linear group/algebra", for groups, this will be the group of all invertible matrices, or, more conceptually and more generally, the group of all invertible linear operators on a vector space.

There are various questions about representations of a given, say, group, one can ask. The most basic question is to classify all possible representations (up to an isomorphism). This is possible in some cases, but not in general. A more narrow question, which the first one sometimes reduces to, is to classify all *irreducible* representations. If all representations (at least in a given class) are completely reducible (a.k.a. semisimple), then one can classify arbitrary representations in terms of irreducible ones. If not, then the full classification is rarely available, but one arrives at questions interesting from Category theory or Homological algebra perspectives. Yet another important question is to get the numerical information about interesting (e.g., irreducible) representations. We can ask about dimensions (for finite dimensional representations) or about finer information – suitably understood characters.

In this class we will care about representations of groups that are close to being simple and related associative and Lie algebras: the symmetric groups, the simple algebraic groups and Lie algebras, finite groups of Lie type, Hecke algebras of various sorts. We will cover classical and more modern (80's and 90's) results and also mention some recent developments.

Various features of this class include:
– this is not a basic introduction to Representation theory, see the list of prerequisites below;
– I plan to post lecture notes;
– we study the representation theory both in characteristic 0 – more classical – and in positive characteristic, the subject of great current interest;
– Hecke algebras make appearance throughout the course;
– essentially, no quantum groups this time;
– we will have some discussion of "categorification";
– if time permits, in the end of the class we will also have some brief discussion of the geometric representation theory, the central branch of the subject for the last 40 years or so. This will require higher prerequisites than the rest of the class (some familiarity with Algebraic topology);
– the course will have a bunch of asides/ digressions. If time permits, they will be discussed in class or added as bonuses to the lecture notes.

2. Description of content

I plan to cover five topics:

- The representation theory of symmetric groups following Okounkov and Vershik.
- The representation theory of semisimple algebraic groups and their Lie algebras in zero and positive characteristic.
- The representation theory of finite groups of Lie type and Hecke algebras.
- The category $\mathcal{O}$ of representations of semisimple Lie algebras, Kazhdan-Lusztig bases and the Soergel theory.
- Time permitting, a brief introduction to Geometric representation theory.

The longest topics are 2 and 4, and 1 and 5 are the shortest.

Below I elaborate on each of these topics. The readers should not be discouraged by the unfamiliar terminology, it's going to be explained in the class. To make the exposition more accessible we will mostly concentrate on "type A" objects: the groups $\mathrm{GL}_n(\mathbb{F})$ or $\mathrm{SL}_n(\mathbb{F})$, the Lie algebras $\mathfrak{sl}_n(\mathbb{F})$, the symmetric group and its Hecke algebra. We will also briefly discuss generalizations to other types.

2.1. **Representations of symmetric groups.** The representation theory of symmetric groups S_n is a basic, yet extremely important, part of the representation theory of finite groups with rich and profound connections to Combinatorics and Geometry. The classical approach dates back to the 30's. In the mid 90's Okounkov and Vershik, [OV], discovered a new approach, which is inductive in nature and is based on the chain of inclusions $S_1 \subset S_2 \subset \ldots \subset S_{n-1} \subset S_n$. Two notable features of this approach are as follows. First, the *degenerate affine Hecke algebras*, that were first introduced in a different and far less elementary context, naturally appear. Second, the Okounkov-Vershik approach can be viewed as one of the predecessors of the theory of "categorical representations" of Lie algebras, the hot topic in Representation theory – and Knot theory – in the 2000's. We may briefly discuss some features of this theory in the second part of the class.

In the first part of the course we will explain the classification of irreducible representations and a description of their bases following Okounkov and Vershik. Along the way, we will recall various basics about representations of finite groups and finite dimensional semisimple algebras. We will also spend a bit of time discussing which parts of the approach survives when the base field has positive characteristic. This case is of great current interest, where even some basic questions are open.

2.2. **Representations of semisimple algebraic groups and Lie algebras.** This is the largest of the five parts of the course. Let $\mathbb{F}$ be an algebraically closed field. By an algebraic group, G, we mean a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{F})$ given by polynomial equations. Example include $\mathrm{SL}_n(\mathbb{F})$, the group of matrices with determinant 1, $\mathrm{O}_n(\mathbb{F})$, the group of orthogonal matrices, and others. Every algebraic group is an affine algebraic variety, and algebraic groups in the context of affine varieties play the role similar to Lie groups in the context of smooth manifolds, in fact, if $\mathbb{F} = \mathbb{C}$, every algebraic group is a complex Lie groups. We will care about "algebraic" representations of G, i.e., representations whose matrix coefficients are polynomial functions on G (e.g., for $G = \mathrm{GL}_n(\mathbb{F})$ this means polynomials in the n^2 matrix entries and $\det^{-1}$).

The tangent space at the unit element to G has an additional structure: it is a Lie algebra – a vector space equipped with a bilinear operation $[\cdot,\cdot]$ called a Lie bracket. For example, for $\mathrm{GL}_n(\mathbb{F})$ we get the space of matrices with matrix commutator, $[x,y] = xy - yx$, denoted in this context by $\mathfrak{gl}_n(\mathbb{F})$ and called the general linear Lie algebra. One can talk about representations of Lie algebras: the finite dimensional ones are, essentially, homomorphisms to $\mathfrak{gl}_n(\mathbb{F})$.

Our goal for this part is to understand the structure and representations of certain algebraic groups and their Lie algebras both when $\mathbb{F}$ has characteristic 0 (classical) and when $\mathbb{F}$ has characteristic p (very hard and of great current interest). Here is a more detailed description.

2.2.1. *Characteristic* 0 *story.* We will start with understanding the finite dimensional representations of the Lie algebra algebraic group $\mathrm{SL}_2(\mathbb{C})$ denoted by $\mathfrak{sl}_2(\mathbb{C})$. The classification is nice and elegant, and, besides, is a building block for understanding the representations (and the structure) of more general Lie algebras. We will then proceed to classifying the representations of $\mathfrak{sl}_n(\mathbb{C})$. Along the way, we will introduce the universal enveloping algebra, the associative algebra controlling the representations of the corresponding Lie algebra in the same way the group algebra controls representations of a group, and describe its basis. We will finish this part by relating the representations of $\mathfrak{sl}_n(\mathbb{C})$ to those of $\mathrm{SL}_n(\mathbb{C})$: they are essentially the same thing. Then we will briefly discuss the Schur-Weyl duality connecting the representations of $\mathrm{SL}_n(\mathbb{C})$ to the those of the symmetric groups.

Next, we will discuss generalizations: the so called (semi)simple Lie algebras and algebraic groups. We will explain (mostly without proof) their classification, structure, and representation theory. Then we will mention a further generalization: Kac-Moody Lie algebras. We may briefly explain how the affine type A Kac-Moody algebras are relevant to the representation theory of symmetric groups, thus connecting this topic to the previous one.

As an aside, we can also discuss the classification of semisimple Lie algebras over the reals as well as the classification of semisimple Lie groups. The point: for non-algebraically closed fields this classification is harder.

2.2.2. *Characteristic p story.* When $\mathbb{F}$ has characteristic $p>0$, the story becomes much more complicated. In short, the classification of semisimple algebraic groups is still the same, but their representation theory as well as the representation theory of the corresponding Lie algebras (unlike in characteristic 0, the connection between the two is quite loose) is not completely understood at this point. These questions were of primary importance for the subject of Representation theory in the last decade.

The bulk of this part will include studying the representations of $\mathrm{SL}_2(\mathbb{F})$ and $\mathfrak{sl}_2(\mathbb{F})$, where, essentially, everything is known, but is considerably more complicated than in characteristic 0. We will finish by a brief discussion of the case of more general semisimple groups such as $\mathrm{SL}_n(\mathbb{F})$. As in the case of symmetric groups even basic questions remain open. We may return to the Lie algebra case when and if we discuss the geometric representation theory.

2.3. **Representations of finite groups of Lie type and Hecke algebras.** Most of finite simple groups are finite groups of Lie type, i.e., modulo technicalities, the groups of $\mathbb{F}_q$-points of algebraic groups over $\overline{\mathbb{F}}_p$. The simplest (and of primary interest for us) example of a finite group of Lie type is $\mathrm{GL}_n(\mathbb{F}_q)$. Our goal is to understand some (in a way, most interesting) irreducible representations of $G_q := \mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$. These are representations that have a vector fixed by the "Borel" subgroup $B_q \subset G_q$ consisting of all upper-triangular matrices. We will show that these irreducible representations are in bijection with those of a certain finite dimensional associative algebra $\mathcal{H}_q(S_n)$ called the Hecke algebra of S_n. It is a deformation of the group algebra $\mathbb{C}S_n$ depending on q (as a parameter). Hecke algebras are extremely important for Representation theory. It can be shown that the irreducuble representations of $\mathcal{H}_q(S_n)$ are naturally labelled by the partitions of n as long as q is not a root of unity of order at most n: the Hecke algebra is actually isomorphic to $\mathbb{C}S_n$, some preferred isomorphisms were discovered in the context of studying categorical representations of Lie algebras. For roots of unity of order $\leqslant n$, the representation theory of $\mathcal{H}_q(n)$ resembles that of symmetric groups over positive characteristic fields.

Then we will explain generalizations to other finite groups of Lie type (e.g., finite orthogonal, symplectic or unitary groups). Possible asides include

- a discussion of the classification of all irreducible representations for finite groups of Lie type,
- a discussion of applications of Hecke algebras to knot invariants,
- the affine Hecke algebras and their connection to the representations of p-adic groups.

2.4. **Category $\mathcal{O}$, etc.** So far, in the context of Lie algebras, we were discussing finite dimensional representations. Infinite dimensional representations of semisimple Lie algebras over $\mathbb{C}$, e.g., of $\mathfrak{sl}_n(\mathbb{C})$, are also very important. The most accessible part of the infinite dimensional representation theory, which also plays an important role in the study of finite dimensional ones are *highest weight modules*. This includes Verma modules. The category $\mathcal{O}$ introduced by Bernstein, I. Gelfand and S. Gelfand is a natural "home" for such modules. While the modules in category $\mathcal{O}$ are infinite dimensional, it still makes sense to speak about their characters. The fact of life is that these characters cannot be expressed using the elementary enumerative combinatorics. The character formulas were conjectured by Kazhdan and Lusztig in 1979 and proved shortly thereafter by Beilinson and Bernstein and by Brylinski and Kashiwara in what was (and still is) a crowning achievement of Geometric representation theory. The answer is stated in terms of a new basis in the Hecke algebra, the Kazhdan-Lusztig basis.

The techniques going into the proofs of the Kazhdan-Lusztig conjecture given in the beginning of the 80's were quite involved (in particular, they used things like the Riemann-Hilbert correspondence, perverse sheaves and Deligne's theory of weights). In 1990 Soergel proposed an alternative and easier approach eliminating some (but not all) of these advanced techniques. His approach had a huge impact on Geometric and Categorical representation theory: he introduced an elementary model for the category $\mathcal{O}$ (or, more precisely, its category of projective objects) – the category of Soergel (bi)modules. Soergel modules are certain graded modules over the polynomial algebra ($\mathbb{C}[x_1, \ldots, x_n]$ if our Lie algebra is $\mathfrak{gl}_n$).

In the course we will introduce the objects mentioned above concentrating on the case of $\mathfrak{sl}_n$ and prove basic results about the category $\mathcal{O}$ (such as the classification of its simple objects and the block decomposition). We may discuss projective functors as well. We will also do many explicit computations in the case of $n = 3$.

Possible asides include

- a discussion of combinatorial shadows of the Kazhdan-Lusztig bases (cells, the asymptotic Hecke algebra),
- a discussion of infinite dimensional representations of real semisimple Lie groups and their connection to "Harish-Chandra modules", certain infinite dimensional representations of Lie algebras.

2.5. **Glimpses of Geometric representation theory.** Geometric representation theory is an interface of Representation theory with Algebraic geometry/topology, it seeks to understand representations using geometric and topological techniques. It has been a central part of the subject for the last 40 years or so. We will introduce two kind of subvarieties in the variety $\mathcal{F}l_n$ of complete flags of vector subspaces in $\mathbb{F}^n$ that play a very important role in the subject:

- Schubert varieties and Bott-Samelson varieties, related to Soergel modules.
- Springer fibers, related to representations of the symmetric group, affine Hecke algebras, Lie algebras in positive characteristic.

We will also discuss the role they play in Representation theory.

3. Prerequisites

There are two primary prerequisites and some secondary (marked with "*").

1) Linear and multilinear algebra: linear operators and bilinear (mostly symmetric) forms, mostly over $\mathbb{C}$. Tensor products and related constructions. This is covered in MATH 240 (the version of Spring 2021) [V, Chapters 5,6,8], or [L, Chapters 13-16].

2) Representation theory: basic notions, constructions and results on representations of finite groups over $\mathbb{C}$ and of finite dimensional semisimple associative algebras (over an algebraically closed field). This is covered in MATH 353/533, or [E, Sections 2-4], [L, Chapters 17,18], [V, Chapter 11] (any of these sources should be close to sufficient). Also a write-up with a reminder will be posted.

3*) Some categorical constructions studied, e.g., in MATH 380/500. This is needed, e.g., for the category $\mathcal{O}$.

4*) Some Algebraic geometry, mostly covered in MATH 380/500 – for the discussion of algebraic groups and also in the discussion of Geometric representation theory.

5*) Some superficial familiarity with Algebraic topology ((co)homology and its basic properties) will be useful for the discussion of geometric Representation theory.

4. Grading and office hours

There will be about five homeworks on topics 1-4. As usual with my courses, they will have more points than the formal maximum. Students will need to get about 60 percent of the total points to score the formal maximum. I plan to give final projects, the list is to be compiled by the Spring break. For students who do not wish to do a project, there will be a final take-home exam.

There will be office hours, twice per week.

5. References

So far, in the list of references below there are three prerequisite references, and one original paper. As the class progresses, more references will be added. Besides, I taught a somewhat similar, although more advanced class, some 5 years ago. For the class materials see this link.

References

[E] P. Etingof et al, *Introduction to Representation theory*. Available from the author webpage.
[L] S. Lang, *Algebra*, 3rd edition. GTM 211, Springer.
[OV] A. Okounkov, A. Vershik, *A new approach to representation theory of symmetric groups*. Selecta Math. (N.S.) 2 (1996), no. 4, 581–605.
[V] E. Vinberg, *A course in Algebra*, GSM 14, AMS.