Representations of symmetric groups, I.

1) Introduction/motivation.

2) Centralizer algebras.

1.1) Reps of finite (almost) simple groups.

Big goal: given a finite group, G, understand its representations

Reasons to care: • rep. theory + structure theory = ♡.

• applications.

Which G do we care about?

An answer: simple, (or "almost simple") groups G.

exactly 2 normal subgroups, $\{1\}$ & G; + G is not abelian.

Finite simple groups have been classified:

• Alternating groups $\mathfrak{A}_n = \{\sigma \in S_n \mid sgn(\sigma) = 1\}$, $n \geqslant 5$.

• 26 sporadic groups.

• The majority: finite groups of Lie type, e.g. $SL_n(\mathbb{F}_q)$/center, simple for most (n,q).

1.2) Case of symmetric groups

$\mathfrak{A}_n \triangleleft S_n$ of index 2 - very close

Rep. th'y of S_n is nicer & can recover rep. th'y of $\mathfrak{A}_n$ from that of S_n

Other reasons to care about reps of S_n:

i) Connection to Combinatorics: of partitions & of symmetric polynomials.

ii) Connection to reps of GL_m via Schur-Weyl duality.

iii) Connection to representations of affine Lie algebras.

Mostly care about base field $\mathbb{C}$.

General things about reps of finite group G over $\mathbb{C}$:

i) Rep'n of G = rep'n of group algebra $\mathbb{C}G$.

ii) $\mathbb{C}G$ is semisimple, $\mathbb{C}G = \bigoplus_V \mathrm{End}_{\mathbb{C}}(V)$

sum is over irreducible reps of $\mathbb{C}G$ (up to isomorphism).

So every rep'n of $\mathbb{C}G$ is completely reducible so we only need to understand irreps.

iii) # irreps of G = # conj. classes in G.

For $G = S_n$: conj. classes in $S_n \xleftrightarrow{\sim} \{$partitions of $n\}$

$$\{g\sigma g^{-1}\} \longmapsto (\text{lengths of cycles})$$

e.g $\sigma = (135)(24) \in S_6 \rightsquigarrow (3,2,1)$, partition of 6.

Notation: for conj. classes $(***)(**) \subset S_{m+5}$ (w. m fixed pts)

for partitions: $(n_1, \dots n_k)$ w. $n_1 \geq n_2 \geq \dots \geq n_k$ - partition of $n = \sum n_k$.

or $(m_1^{d_1}, \dots m_\ell^{d_\ell})$ where $m_1 > m_2 > \dots > m_\ell$ & $d_1, \dots d_\ell$ are multiplicities

e.g $(2,2,1,1) = (2^2, 1^2)$ - part'n of 6.

Goal: establish bijection between $\{$partitions of $n\}$ & $\mathrm{Irr}(\mathbb{C}S_n) = \{$isom. classes of irreducible $\mathbb{C}S_n$-modules$\}$.

following Okounkov-Vershik.

Example: for S_4: irreps

$triv_4$	(4)
sgn_4	(1^4)
$refl_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_1+x_2+x_3+x_4=0\}$	$(3,1)$
$sgn_4 \otimes refl_4$	$(2,1^2)$
$\mathbb{C}^2$	$(2,2)$

Rem: for rep'n V of S_n, $V \otimes sgn_n$ is same vector space but the action of each permutation is multiplied by its sign.

2) Centralizer algebra

Basic idea: "induction" $S_1 \subset S_2 \subset S_3 \subset \ldots \subset S_k \subset S_{k+1} \subset \ldots$

$S_k = \{\sigma \in S_{k+1} \mid \sigma(k+1) = k+1\}$

Want: study S_n-irreps by restricting to S_{n-1}

2.1) Centralizer algebra & restriction of reps:

Question: Given $V \in Irr(\mathbb{C}S_n)$, decompose it into $\oplus$ of S_m-irreps $(m<n)$. We'll need $m=n-1$ but also $m=n-2$.

More general: for finite $H \subset G$, finite gr'ps, decompose $V \in Irr(\mathbb{C}G)$ into $\oplus$ of H-irreps.

$\mathbb{C}H \subset \mathbb{C}G$ – semisimple assoc. algebras.

Even more general: given $B \subset A$ fin. dim'l s/simple assoc. alg's & $V \in Irr(A)$ decompose V as $\bigoplus$ of B-irreps.

And yet more general: B, A fin. dim'l s/simple assoc. $\mathbb{C}$-algebras, $\tau: B \to A$ alg. homom., $V \in Irr(A)$ (so also B-module)
Then the same question.

Recall:

$$A \simeq \bigoplus_{V \in Irr(A)} End_{\mathbb{C}}(V) \qquad (1)$$

$$B \simeq \bigoplus_{U \in Irr(B)} End_{\mathbb{C}}(U)$$

For $V \in Irr(A)$, $U \in Irr(B) \rightsquigarrow$ "multiplicity space"

$$M_{V,U} := Hom_B(U, V) \text{ - vector space}$$

$Irr(B) = \{U_1, \dots, U_k\}$

Know:

$$\bigoplus_{i=1}^{k} U_i \otimes_{\mathbb{C}} M_{V,U_i} \xrightarrow{\sim} V, \quad \sum_{i=1}^{k} u_i \otimes \varphi_i \mapsto \sum_{i=1}^{k} \varphi_i(u_i)$$

B-linear

Point: Nonzero spaces $M_{V,U}$ = irreducible reps of a certain algebra

Definition: Centralizer algebra $Z_B(A) = \{a \in A \mid a\tau(b) = \tau(b)a \ \forall b \in B\}$

E.g. $B=A$, $\tau = id$, then $Z_B(A)$ is the center $Z(A)$

Exercise: $Z_B(A) \subset A$ is subalgebra.

Lemma: $\exists$ algebra isom'm $Z_B(A) = \bigoplus End(M_{V,U})$, where $\bigoplus$ is over pairs $V \in Irr(A)$, $U \in Irr(B)$ s.t. $M_{V,U} \neq \{0\}$.

i.e. $Z_B(A)$ is semisimple & its irreps are the nonzero spaces $M_{V,U}$.

Example: $A = Mat_4(\mathbb{C}) \oplus Mat_3(\mathbb{C})$, $B = Mat_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$

$\tau(x_1, x_2, x_3) = (diag(x_1, x_2, x_2), diag(x_1, x_3))$

$\dim M_{V_1,U_2} = 2$, $\dim M_{V_1,U_1}, M_{V_2,U_1}, M_{V_2,U_3} = 1$; $M_{V_1,U_3} = M_{V_2,U_2} = \{0\}$.

Lemma predicts $Z_B(A) = Mat_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}$

Check:

$$Z_B(A) = \{(y_1, y_2) \mid \begin{array}{l} y_1 \in Mat_4(\mathbb{C}) \text{ commutes w. } \forall\, diag(x_1, x_2, x_2) \\ y_2 \in Mat_3(\mathbb{C}) \ - \cdot - \cdot - \cdot \ \forall\, diag(x_1, x_3) \end{array}\}$$

$$= \left\{ \underbrace{\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & d & e \end{pmatrix}}_{y_1}, \underbrace{\begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix}}_{y_2} \;\middle|\; a, b, c, d, e, f \in \mathbb{C} \right\}$$

$Z_B(A) \xrightarrow{\sim} Mat_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}$: $(y_1, y_2) \mapsto \left(\begin{pmatrix} b & c \\ d & e \end{pmatrix}, a, f, g \right)$.

Proof of Lemma: By (1), $A \xrightarrow{\sim} \bigoplus_V \mathrm{End}(V)$

$\tau = (\tau_V)$, where $\tau_V : B \longrightarrow \mathrm{End}(V)$.

$$Z_B(A) = \{(a_V) \mid a_V \in \mathrm{End}(V) \mid a_V \tau_V(b) = \tau_V(b) a_V\}$$

$$= \bigoplus_V Z_B(\mathrm{End}(V)) = \bigoplus_V \mathrm{End}_B(V)$$

$$= [\, V \simeq \bigoplus_U U \otimes M_{V,U} \Rightarrow \mathrm{End}_B(V) \overset{\text{part } 0}{\simeq} \bigoplus_{U \mid M_{V,U} \neq \{0\}} \mathrm{End}(M_{V,U})\,]$$

$$= \bigoplus_{V,U} \mathrm{End}(M_{V,U}). \qquad \square$$

Rem: How to define $Z_B(A)$-module structure on $M_{V,U} = \mathrm{Hom}_B(U,V)$ w/o referring $A \xrightarrow{\sim} \bigoplus_V \mathrm{End}(V)$.

$z \in Z_B(A)$, $\varphi \in \mathrm{Hom}_B(U,V)$

$[z\varphi](u) = z[\varphi(u)]$

Exercise (or see Rem 2.5 in RT1): this is the module structure implied by lemma.

Corollary (of Lemma): TFAE

(a) $\forall\ V \in \mathrm{Irr}(A),\ U \in \mathrm{Irr}(B)$, $\dim M_{V,U} \leq 1$

(b) $Z_B(A)$ is commutative

Proof: By Lemma, $Z_B(A) \simeq \bigoplus_{V,U} \mathrm{End}(M_{V,U})$ is comm've $\Leftrightarrow$ $\mathrm{End}(M_{V,U})$ is comm've $\forall\ V,U \Leftrightarrow \dim M_{V,U} \leq 1\ \forall\ U,V$. $\square$

Under assump's of Cor have $V = \bigoplus_{U \mid \dim M_{V,U} = 1} U$ - decomposition into $\bigoplus$ of B-submodules, where U is uniquely determined as the image of <ins>any</ins> nonzero el't of $Hom_B(U, V)$.