Lecture 10.66: Frobenius kernel.

Depending on Lectures 6.5, 8.5, and a bit on 10.33.

The goal of this note is to elaborate on the Frobenius homomorphism $Fr: G \longrightarrow G$ & on the connection between representations of G & its Lie algebra $\mathfrak{g}$ in characteristic p. In short:

- Fr has kernel, the non-reduced group subscheme $G_1 \subset G$ w. a single point.
- As a Hopf algebra, $Dist_1(G) = \mathbb{F}[G_1]^*$ is $U^0\mathfrak{g}$, the p-central reduction (see the complement to Lec 9).

This is why the representation of $\mathfrak{g}$ arising from a rational G-representation factors through $U^0\mathfrak{g}$.

We will elaborate on these claims below.

1) Frobenius twist.

We want to understand the Frobenius morphism more conceptually.

Definition: For a vector space V over $\mathbb{F}$, define a new vector space, $V^{(1)}$, that is identified w. V as an abelian group but the action of $\mathbb{F}$ is twisted by Fr^{-1}.

Exercise: how does this relate to the construction of Frobenius twist for rational representations in Sec 2 of Lec 9?

Note that $V^{(1)}$ inherits algebraic structures from V: e.g. an associative $\mathbb{F}$-algebra structure on V remains an associative $\mathbb{F}$-algebra structure on $V^{(1)}$.

Here's the reason to make this definition: let A be a commutative $\mathbb{F}$-algebra. Then $a \mapsto a^p$ is an $\mathbb{F}$-algebra (not just a ring homomorphism) $A^{(1)} \to A$.

Exercise: if $A \xrightarrow{\sim} \mathbb{F} \otimes_{\mathbb{F}_p} \underline{A}$ for an $\mathbb{F}_p$-algebra $\underline{A}$, then $A^{(1)} \simeq A$.

Now suppose that X is an affine variety. By $X^{(1)}$ we mean the variety corresponding to $\mathbb{F}[X]^{(1)}$. If $X \subset \mathbb{F}^n$ is given by some polynomial equations, then $X^{(1)} \subset \mathbb{F}^{n(1)}$ is given by the same equations so in $\mathbb{F}^n$, $X^{(1)}$ would be given by equations whose coefficients are twisted by $\alpha \mapsto \alpha^p$!

Note that the homomorphism $A^{(1)} \to A$, $a \mapsto a^p$, gives rise to a morphism of varieties $X \to X^{(1)}$ denoted by Fr. For example, if we identify $\mathbb{F}^{n(1)}$ w. $\mathbb{F}^n$ (see the previous exercise), then $Fr: \mathbb{F}^n \to \mathbb{F}^{n(1)}$ is just $(x_1, \dots, x_n) \mapsto (x_1^p, \dots, x_n^p)$

Exercise: if G is an algebraic group, then so is $G^{(1)}$ & $Fr: G \to G^{(1)}$ is a group homomorphism.

If $G \subset GL_n(\mathbb{F})$ is defined by polynomial equations w. coefficients in $\mathbb{F}_p$, then $G \simeq G^{(1)}$ and Fr is the homomorphism constructed in Sec 1.2 of Lec 5.

2) Frobenius kernel.

An algebraic group G is a variety so $\mathbb{F}[G]$ has no nilpotent elements. Then $Fr^*: f \mapsto f^p: \mathbb{F}[G^{(1)}] \to \mathbb{F}[G]$ is injective. So Fr is dominant, hence (it's a group homom'm) surjective. In this situation we are supposed to have $G^{(1)} \simeq G/\ker Fr$.

While Fr is a bijective homomomomorphism of abstract groups, its <u>scheme theoretic</u> fibers are nontrivial. We'll be interested in $\ker Fr = Fr^{-1}(1)$. Let $\mathfrak{m} \subset \mathbb{F}[G^{(1)}]$ be the maximal ideal. Then the algebra of functions on $Fr^{-1}(1)$ (as a scheme) is $\mathbb{F}[Fr^{-1}(1)] := \mathbb{F}[G]/\mathbb{F}[G]\{f^p \mid f \in \mathfrak{m}\}$. Here is an important

Exercise: $\mathbb{F}[G]\{f^p \mid f \in \mathfrak{m}\}$ is a "Hopf ideal" meaning there is a unique Hopf algebra structure on $\mathbb{F}[Fr^{-1}(1)]$ s.t. the projection $\mathbb{F}[G] \twoheadrightarrow \mathbb{F}[Fr^{-1}(1)]$ is a Hopf algebra homomorphism.

So $Fr^{-1}(1)$ is a kind of a group, more precisely, it's a "group scheme" (take the conceptual definition in Remark in Sec 1.1 and replace varieties with (finite type) schemes).

We write $G_1 = \ker Fr$. Note that the inclusion $G_1 \subset G$ gives rise to $\mathbb{F}[G_1]^* = Dist_1(G_1) \hookrightarrow Dist_1(G)$. This is an inclusion of

Hopf algebras.

Consider the examples from Sec 2.2. of Lec 6.5.

Example 1: $G=\mathbb{G}_a$, the additive group. Then $\mathbb{F}[G_1]=\mathbb{F}[x]/(x^p)$ w. $\Delta(x)=x\otimes 1+1\otimes x$. This Hopf algebra is actually isomorphic to its dual, $Dist_1(G_1)$.

Example 2: $G=\mathbb{G}_m$, the multiplicative group. Then $\mathbb{F}[G_1]=\mathbb{F}[x]/((x-1)^p)$. The algebra $Dist_1(G_1)$ has basis $\delta_0,\delta_1,\ldots\delta_{p-1}$, where $\delta_i:=\binom{\delta}{i}$ for $\delta=\delta_1$. And $\delta(\delta-1)\ldots(\delta-(p-1))=0$ in $Dist_1(G)$, hence $Dist_1(G_1)=\mathbb{F}[\delta]/(\delta^p-\delta)$, as an algebra, with $\Delta(\delta)=\delta\otimes 1+1\otimes\delta$.

3) $Dist_1(G_1)$ vs $\mathcal{U}(\mathfrak{g})$.

Recall, Section 1 of Lec 8.5, that we have an algebra homomorphism $\mathcal{U}(\mathfrak{g})\longrightarrow Dist_1(G)$ (in fact, a homomorphism of Hopf algebras). The elements of $\mathfrak{g}$ annihilate functions of the form f^p hence $\mathbb{F}[G]\{f^p\}$, so, the image of $\mathcal{U}(\mathfrak{g})$ is $Dist_1(G_1)$.

Lemma: For an $\xi\in\mathfrak{g}$, the distributions given by ξ^p & $\xi^{[p]}$ coincide.

Sketch of proof: The distribution given by a monomial $\xi_1\ldots\xi_k\in\mathcal{U}(\mathfrak{g})$ w. $\xi_1,\ldots\xi_k\in\mathfrak{g}$ can be shown to be $f\mapsto[\tilde{\xi}_1\ldots\tilde{\xi}_k f](1)$, where we write $\tilde{\xi}_i$ for the element of $Vect(G)^G$ corresponding to $\xi_i\in\mathfrak{g}$. Now it remains to notice that $\tilde{\xi}^p f=\tilde{\xi}^{[p]}f\ \forall f\in\mathbb{F}[G]$, see the discussion in the complement section of Lec 10. $\square$

So, $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{Dist}_1(G_1)$ factors through $\mathcal{U}^0(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/(\xi^p - \xi^{[p]} \mid \xi \in \mathfrak{g})$.

Thm: $\mathcal{U}^0(\mathfrak{g}) \longrightarrow \mathrm{Dist}_1(G_1)$ is an isomorphism (of Hopf algebras).
Note that this theorem can be viewed as an analog of Thm in Sec 1 of Lec 8.5.

Sketch of proof: As a variety, G is smooth. Let's say $\dim G = n$. We can pick a so called "etale coordinate chart" $x_1, \dots, x_n$ at 1: $x_1, \dots, x_n \in \mathbb{F}[G]$ s.t. for $F = (x_1, \dots, x_n): G \longrightarrow \mathbb{F}^n$ we have that $T_1 F$ is an isomorphism. We can lift the partials from $\mathbb{F}^n$ to a neighborhood of 1 in G, denote the resulting local vector fields on G by $\partial_1, \dots \partial_n$. By the construction, their values at $1 \in G$ form a basis in $T_1 G$. Denote them by $\xi_1, \dots, \xi_n$. Then one shows that:

(i) $\mathbb{F}[x_1, \dots x_n]/(x_1^p, \dots, x_n^p) \xrightarrow{\sim} \mathbb{F}[G_1]$

(ii) $\mathbb{F}[\partial_1, \dots \partial_n]/(\partial_1^p, \dots, \partial_n^p) \xrightarrow{\sim} \mathrm{Dist}_1(G_1)$

(iii) $\tilde{\xi}_i = \partial_i + \sum_{j=1}^n f_{ij} \partial_j$ w. $f_{ij}(1) = 0$.

The claim that $\mathcal{U}^0 \mathfrak{g} \xrightarrow{\sim} \mathrm{Dist}_1(G)$ follows from here: the elements $\xi_1^{d_1} \cdots \xi_n^{d_n}$ w. $0 \leq d_1, \dots, d_n \leq p-1$, form a basis in $\mathcal{U}^0 \mathfrak{g}$ and the image of $\xi_1^{d_1} \cdots \xi_n^{d_n}$ in $\mathrm{Dist}_1(G_1)$ has the form $\partial_1^{d_1} \dots \partial_n^{d_n} + $ l.d.t. (lower degree terms). $\square$

Example: Let $G = \mathbb{G}_m = GL_1(\mathbb{F})$. Let $\xi \in \mathfrak{g} = \mathfrak{gl}_1(\mathbb{F})$ be the element corresponding to 1. Then $\xi^{[p]}=\xi$ & $\mathcal{U}^0(\mathfrak{g}) = \mathbb{F}[\xi]/(\xi^p-\xi)$. This algebra is isomorphic to $Dist_1(G_1)$ by Example 2 in Sec 2, and, indeed, the homomorphism $\mathcal{U}^0(\mathfrak{g}) \longrightarrow Dist_1(G_1)$ sends ξ to δ.

4) Rational representations of G w. trivial $\mathfrak{g}$-action.

This section addresses the comment after Problem 4 in HW2.

Let V be a rational representation of G. We can restrict it to a rational representation of G_1. By Section 3 of Lec 8.5, a rational representation of G_1 is the same thing as an $\mathbb{F}[G_1]$-comodule, which is the same thing as a $Dist_1(G_1)$-module.

Further, we can view V as a $Dist_1(G)$-module. By the construction of the latter, the $\mathcal{U}(\mathfrak{g})$-action on V is obtained from the $Dist_1(G)$-action via the homomorphism $\mathcal{U}(\mathfrak{g}) \to Dist_1(G)$ from Sec 1 of Lec 8.5 (exercise). So this action factors through $\mathcal{U}(\mathfrak{g}) \twoheadrightarrow \mathcal{U}^0(\mathfrak{g}) \xrightarrow{\sim} Dist_1(G_1)$. In other words, passing from a representation of an algebraic group to the representation of its Lie algebras is equivalent to restriction to G_1.

In particular, $\mathfrak{g}$ acts trivially on $V \Leftrightarrow G_1$ acts trivially $\Leftrightarrow V$ is obtained as the pullback of a rational representation of $G^{(1)} \simeq G/G_1$, i.e., V arises as the Frobenius twist of some rational representation of $G^{(1)}$.