Representations of algebraic groups & Lie algebras, VI

1) Representations of $\mathfrak{sl}_2(\mathbb{F})$, char $\mathbb{F}>2$.
2) Representations of $SL_2(\mathbb{F})$, char $\mathbb{F}>2$.
3) Complements.

1.0) Recap: Let $\mathbb{F}$ be an algebraically closed field of char $=p>2$. Set $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{F})$. We've seen in Sec 3 of Lec 9 that the elements $e^p, h^p-h, f^p \in \mathcal{U}(\mathfrak{g})$ are central. We've also classified the $\mathfrak{g}$-irreps where these elements act by $\chi=$

$$(0,0,0)$$
$$(0,0,1)$$
$$(0,a,0),\ a\neq 0.$$

Our goal in this part is to explain how the three central elements arise and also explain, why it's sufficient to classify the irreps corresponding to the triples above - by the complement section in Lec 9, every triple can occur.

1.1) Restricted p-th power map

This is an additional structure of Lie algebras of algebraic groups in characteristic p. Let $G\subset GL_n(\mathbb{F})$ be an algebraic group.

Theorem: 1) $\mathfrak{g}\subset\mathfrak{gl}_n(\mathbb{F})\ (=\mathrm{Mat}_n(\mathbb{F}))$ is closed under $x\mapsto x^p$. We will use the notation $x^{[p]}$ for x^p in this context.

2) Let $\Phi: G\to H$ be an algebraic group homomorphism, and $\varphi=T_1\Phi$:

$\mathfrak{g} \to \mathfrak{h}$ the induced Lie algebra homomorphism. Then $\varphi(x^{[p]}) = \varphi(x)^{[p]}$

Note that 2) shows that $x \mapsto x^{[p]}$ is recovered from G itself and not from an embedding $G \hookrightarrow GL_n(\mathbb{F})$.

Exercise: Check 1) for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F}), \mathfrak{sp}_n(\mathbb{F})$.

Rem: This theorem is parallel to Thm in Sec 2 of Lec 6. The proof is morally similar: we "recover" $x \mapsto x^{[p]}: \mathfrak{g} \to \mathfrak{g}$ from $g \mapsto g^p: G \to G$. This requires the language of groups of points over truncated polynomial rings, see the complement section and compare to the complement to Lec 6. A key computation is that for a curve of the form $g(t) = 1 + t(\xi + t\ldots)$ in $GL_n(\mathbb{F})$ we have $g(t)^p = 1 + t^p \xi^p + t^{p+1}\ldots$ (for two commuting elements α, β in any associative $\mathbb{F}$-algebra we have

$$(\alpha+\beta)^p = \alpha^p + \beta^p. \qquad (1)$$

Example: for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$, we have $e^{[p]} = f^{[p]} = 0$, $h^{[p]} = h$.

1.2) The p-central map $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$.

Let $\mathfrak{g}$ be the Lie algebra of an algebraic group G. For $x \in \mathfrak{g}$ can consider $x^p \in \mathcal{U}(\mathfrak{g})$ (of degree p) & $x^{[p]} \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$. So we have a map $\iota: x \mapsto x^p - x^{[p]}: \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$.

To state one of its properties, we need a general construction (over any $\mathbb{F}$). Recall (Lec 7, Sec 1.3) the adjoint representation $Ad: G \to GL(\mathfrak{g})$: $Ad(g) = T_1 a_g$, where $a_g: g' \mapsto gg'g^{-1}: G \to G$, a group automorphism. T_1 of a group homomorphism is a Lie algebra homomorphism, Sec 2 of Lec 6. So $Ad(g)$ is a Lie algebra automorphism.

Any Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}$ extends to an associative algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$.

So we get the adjoint actions of G by Lie algebra automorphisms on $\mathfrak{g}$ and by associative algebra automorphisms on $\mathcal{U}(\mathfrak{g})$.

Now we get back to $\operatorname{char} \mathbb{F} = p > 2$.

Theorem: 1) $\iota(x)$ is central $\forall x \in \mathfrak{g}$.

2) $\iota(ax) = a^p \iota(x)$ $\forall a \in \mathbb{F}, x \in \mathfrak{g}$.

3) $\iota(x+y) = \iota(x) + \iota(y)$, $\forall x, y \in \mathfrak{g}$.

4) The map ι is G-equivariant: $\iota(Ad(g)x) = Ad(g)\iota(x)$, $\forall g \in G, x \in X$.

Example: for $\mathfrak{g} = \mathfrak{sl}_2$, have $\iota(e) = e^p$, $\iota(h) = h^p - h$, $\iota(f) = f^p$

Proof of Theorem: We'll prove 1) in detail.

Claim: Let A be an associative $\mathbb{F}$-algebra. For $x \in A$, we write $ad(x)$ for the linear operator $y \mapsto [x, y]: A \to A$. Then $x^p y - y x^p = ad(x)^p y$.

Proof of claim: Let L_x, R_x be the operators $y \mapsto xy$, $y \mapsto yx$: $A \to A$. Note that L_x, R_x commute & $ad(x) = L_x - R_x$. So $ad(x)^p = (L_x - R_x)^p = [(1)] = L_x^p - R_x^p = L_{x^p} - R_{x^p} = ad(x^p)$. $\square$ of claim.

Now we get back to proving 1). Apply Claim to $A=\mathcal{U}(\mathfrak{g})$
$x,y\in\mathfrak{g}\subset\mathcal{U}(\mathfrak{g}) \rightsquigarrow [x^p,y]=\operatorname{ad}(x)^p y$. Note that all operations in the right hand side in $\mathfrak{g}$. Now apply Claim to $A=\operatorname{Mat}_n(\mathbb{F})$ (used to define $x^{[p]}$), get $[x^{[p]},y]=\operatorname{ad}(x)^p y$. So $[x^p-x^{[p]},y]$ $=\operatorname{ad}(x)^p y-\operatorname{ad}(x)^p y=0$ $\forall y\in\mathfrak{g}$. Hence $c(x)=x^p-x^{[p]}$ is central.

3) is similar in spirit but much harder (see Lec 10.33): for $x,y\in A$ $(x+y)^p-x^p-y^p$ is a "universal" (independent of choices of A,x,y) Lie polynomial (= expression in brackets) in x,y.

2) and 4) are easy and left as exercises. $\square$

1.3) Completion of classification.

Let's explain how the theorem helps in classifying the irreducible representations of $\mathfrak{g}$. Pick an irreducible $\mathfrak{g}$-representation (= $\mathcal{U}(\mathfrak{g})$-module) V. For $x\in\mathfrak{g}$, let $\chi_V(x)$ denote the scalar of the action of the central element $c(x)\in\mathcal{U}(\mathfrak{g})$ on V, to be called the p-central character of V.

We will also need a basic tool to produce new representations from existing ones. For $g\in G$ define a representation V^g of $\mathfrak{g}$ as follows: if $\mathfrak{g}$ acts on V via a homomorphism $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$, then $\mathfrak{g}$ acts on V^g via $\rho\circ\operatorname{Ad}(g)$. By 4) of Theorem, we have (exercise):

$$\chi_V\circ\operatorname{Ad}(g)=\chi_{V^g} \qquad (*)$$

We write $\mathfrak{g}^{*(1)}$ for the set of functions $\chi:\mathfrak{g}\to\mathbb{F}$ satisfying $\chi(x+y)=\chi(x)+\chi(y)$, $\chi(ax)=a^p\chi(x)$ $\forall x,y\in\mathfrak{g}, a\in\mathbb{F}$. Thx to (2) & (3) of

Thm, and the fact that χ_V is a linear functon on the center we have $\chi_V \in \mathfrak{g}^{*(1)}$ $\forall$ $\mathfrak{g}$-irreps V.

Remark: The group G acts on $\mathfrak{g}^{*(1)}$ by $g.\chi := \chi \circ Ad(g^{-1})$. Thx to (*) it's enough to classify irreps for just one χ_V per orbit: $V \mapsto V^g$ gives a bijection between irreps w. p-central characters χ and $\chi \circ Ad(g)$.

The next exercise describes the G-action on $\mathfrak{g}^{*(1)}$: it's "essentially" the adjoint action on $\mathfrak{g}$:

Exercise: 1) Show that $(x,y) := t(xy)$ defines a G-invariant non-degenerate symmetric form on $\mathfrak{g}$.

2) Show that for $\chi \in \mathfrak{g}^{*(1)}$ $\exists!$ $z_\chi \in \mathfrak{g}$ s.t. $\chi(x) = tr(z_\chi Fr(x))$ $\forall x \in \mathfrak{g}$, where $Fr\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$. The map $\chi \mapsto z_\chi$ is G-equivariant: $\chi \circ Ad(g^{-1}) \mapsto Fr(g) z_\chi Fr(g)^{-1}$

Example: $\chi = (0,0,0), (0,0,1), (0,a,0)$ have $z_\chi = 0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a/2 & 0 \\ 0 & -a/2 \end{pmatrix}$.

These z_χ's are exactly representatives of all G-orbits (= conjugacy classes) in $\mathfrak{g}$ by the JNF theorem. This together with the remark before the exercise finishes the classification of finite dimensional irreducible representations of $\mathfrak{g}$.

2) Representations of $SL_2(\mathbb{F})$ w. char $\mathbb{F} > 2$.

Our task is to classify the irreducible rational representations. The $M(i)$'s are still there but generally are no longer irreducible.

Example: $M(i) = \mathrm{Span}_{\mathbb{F}}(x^i, x^{i-1}y, \dots, y^i)$ is

- for $i<p$, irreducible over $\mathfrak{g}$ (Sec. 3 of Lec 9) and so, since every G-subrep. is also $\mathfrak{g}$-subrep, over G.
- $M(p)$ is not irreducible: $\mathrm{Span}_{\mathbb{F}}(x^p, y^p)$ is a submodule: e.g. for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, have $g.x^p = (ax+cy)^p = a^p x^p + c^p y^p \in \mathrm{Span}_{\mathbb{F}}(x^p, y^p)$.

In fact, "most" of $M(i)$'s w. $i \geqslant p$ are not completely reducible.

Now we produce more irreducible objects. For this we need the "Frobenius twist" construction.

Definition: Let V be a rational representation of G and $\rho: G \to GL(V)$ be the corresponding homomorphism. The Frobenius twist, $V^{(1)}$, is the representation corresponding to $\rho \circ Fr: G \to GL(V)$, where, recall, $Fr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$, is the Frobenius homomorphism (Ex 2 in Sec 1.2 of Lec 5).

Example: $\mathrm{Span}_{\mathbb{F}}(x^p, y^p) \simeq M(1)^{(1)}$.

Observation: Fr is an abstract group isomorphism, so $V^{(1)}$ is irreducible $\Leftrightarrow$ V is.

Proposition: If V is irreducible, then $M(i) \otimes V^{(1)}$ is irreducible $\forall i \in \{0, \dots, p-1\}$.

Proof: Note that $\mathfrak{g}$ acts on $V^{(1)}$ by 0 ($T_1 Fr = 0$). So any $\mathfrak{g}$-subrepresentation of $M(i) \otimes V^{(1)}$ is of the form $M(i) \otimes V'$ for a subspace $V' \subset V^{(1)}$.

Note that $g(M^{(i)} \otimes V') = M^{(i)} \otimes gV'$. So $M^{(i)} \otimes V'$ is G-stable $\Leftrightarrow$ $V' \subset V^{(1)}$ is G-stable. Since V (hence $V^{(1)}$) is irreducible, we are done. $\square$

This gives rise to the following inductive construction. For $k>0$, we write $\bullet^{(k)}$ for $\bullet^{(1)}$ repeated k times.

Corollary (Steinberg tensor product theorem) For $0 \leq \lambda_0, \ldots, \lambda_k \leq p-1$, the representation $M(\lambda_0) \otimes M(\lambda_1)^{(1)} \otimes \ldots \otimes M(\lambda_k)^{(k)}$ is irreducible.

In fact, we'll see that these modules are pairwise non-isomorphic and exhaust all irreducible rational representations of $SL_2(\mathbb{F})$.

3) Complements: conceptual description of $x \mapsto x^{[p]}$

This part depends on the complement to Lecture 6.

• Via points of $A_i := \mathbb{F}[\varepsilon]/(\varepsilon^i)$: consider the group $G(A_i)$ (see the complement to Lec 6). Let $G_1(A_{p+1})$ be the kernel of $G(A_{p+1}) \to G$ & $G_p(A_{p+1})$ be the kernel of $G(A_{p+1}) \longrightarrow G(A_p)$, the latter is identified w. $\mathfrak{g}$. Consider the map $g \mapsto g^p : G(A_{p+1}) \longrightarrow G(A_{p+1})$. Extending the computation in Remark of Section 1.1, we see that this map restricts to $G_1(A_{p+1}) \longrightarrow G_p(A_{p+1})$ and moreover, factors through $\mathfrak{g} = G_1(A_{p+1})/G_2(A_{p+1}) \longrightarrow G_p(A_{p+1})$. Theorem in Section 1.1 follows.

• Via invariant vector fields: an observation is that for a commutative algebra A & a derivation $\delta: A \to A$, the map $\delta^p : A \to A$. For $A = \mathbb{F}[G]$ & left-invariant vector fields, the map $\delta \mapsto \delta^p$ turns out to coincide w $\cdot^{[p]}$. For this one needs to prove that for $G = GL_n(\mathbb{F})$ we recover taking the pth power in $\mathfrak{gl}_n(\mathbb{F}) = Mat_n(\mathbb{F})$ & then prove (2) of Thm.