1) Representations of $\mathfrak{s}_2(F)$, char $F > 2$.

2) Representations of $\mathfrak{s}_2(F)$, char $F > 2$.

3) Complements.

1.0) Recap: Let $F$ be an algebraically closed field of char $= p > 2$. Set $\mathfrak{g}_F = \mathfrak{s}_2(F)$. We've seen in Sec 3 of Lec 9 that the elements $e^p, h^p - h, f^p \in U(\mathfrak{g})$ are central. We've also classified the $\mathfrak{g}$-irreps where these elements act by $X =
\begin{align*}
(0,0,0) \\
(0,1,1) \\
(0,a,0), a \neq 0
\end{align*}$

Our goal in this part is to explain how the three central elements arise and also explain why it's sufficient to classify the irreps corresponding to the triples above - by the complement section in Lec 9, every triple can occur.

1.1) Restricted $p$-th power map

This is an additional structure of Lie algebras of algebraic groups in characteristic $p$. Let $G \subseteq GL_n(F)$ be an algebraic group.

**Theorem:** 1) $g \in \mathfrak{gl}_n(F) (= \text{Mat}_n(F))$ is closed under $x \mapsto x^p$. We will use the notation $x^{p^n}$ for $x^p$ in this context.

2) Let $\varphi: G \to H$ be an algebraic group homomorphism, and $\varphi = T_1 \varphi^p$.
\( g \rightarrow \mathfrak{g} \) the induced Lie algebra homomorphism. Then \( q(x^{[p]}; = q(x)^{[p]} \)

Note that 2) shows that \( x \mapsto x^{[p]} \) is recovered from \( G \) itself and not from an embedding \( G \rightarrow Cl_n(\mathbb{F}) \).

**Exercise:** Check 1) for \( g = SL_n(\mathbb{F}), SO_n(\mathbb{F}), Sp_n(\mathbb{F}) \).

**Rem:** This theorem is parallel to Thm in Sec 2 of Lec 6. The proof is morally similar: we "recover" \( x \mapsto x^{[p]}: \mathfrak{g} \rightarrow \mathfrak{g} \) from \( g \rightarrow g^p: G \rightarrow G \). This requires the language of groups of points over truncated polynomial rings, see the complement section and compare to the complement to Lec 6. A key computation is that for a curve of the form \( g(t) = 1 + t(s + t \ldots) \) in \( GL_n(\mathbb{F}) \) we have \( g(t)^p = 1 + t^p s^p + t^{p+1} \ldots \) (for two commuting elements \( s, s \) in any associative \( \mathbb{F} \)-algebra we have

\[
(d + b)^p = d^p + b^p
\]

**Example:** For \( g = SL_2(\mathbb{F}) \), we have \( e^{[p]} = f^{[p]} = 0, h^{[p]} = h \).

1.2) The \( p \)-central map \( g \rightarrow U(g) \).

Let \( \mathfrak{g} \) be the Lie algebra of an algebraic group \( G \). For \( x \in \mathfrak{g} \) can consider \( x^p \in U(g) \) (of degree \( p \)) & \( x^{[p]} \in \mathfrak{g} \subset U(g) \). So we have a map \( \mathfrak{c}: x \mapsto x^p - x^{[p]}: \mathfrak{g} \rightarrow U(g) \).

To state one of its properties, we need a general construction (over \( \mathbb{F} \)).
any \([F].\) Recall (Lec 7, Sec 13) the adjoint representation \(\text{Ad}: G \rightarrow GL(g):\)
\[ \text{Ad}(g) = T_g, \] where \(g: g' \mapsto gg'^{-1}: G \rightarrow G,\) a group automorphism. \(T_g\) of a
group homomorphism is a Lie algebra homomorphism, Sec 2 of Lec 6.
So \(\text{Ad}(g)\) is a Lie algebra automorphism.

Any Lie algebra homomorphism \(\phi \mapsto \phi\) extends to an associative algebra homomorphism \(U(g) \rightarrow U(g)\).
So we get the adjoint actions of \(G\) by Lie algebra automorphisms on \(g\) and by associative algebra automorphisms on \(U(g)\).

Now we get back to char \([F]=p>2.\)

**Theorem:** 1) \(L(x)\) is central \(\forall x \in g.\)
2) \(L(ax) = axL(x) \forall a \in F, x \in g.\)
3) \(L(x+y) = L(x)+L(y), \forall x, y \in g.\)
4) The map \(L\) is \(G\)-equivariant: \((\text{Ad}(g)L(x)) = \text{Ad}(g)\cdot L(x), \forall g \in G, x \in X.\)

**Example:** for \(g = S^5\), have \(L(e) = e^p, L(h) = h^p - h, L(f) = f^p\)

**Proof of Theorem:** We’ll prove 1) in detail.

**Claim:** Let \(A\) be an associative \([F]\)-algebra. For \(x \in A\), we write
\(ad(x)\) for the linear operator \(y \mapsto [x, y]: A \rightarrow A.\) Then \(x^p y - yx^p = ad(x)^p y.\)

**Proof of claim:** let \(L_x, R_x\) be the operators \(y \mapsto xy, y \mapsto yx: A \rightarrow A.\) Note that \(L_x, R_x\) commute & \(ad(x) = L_x - R_x.\) So
\[ ad(x)^p = (L_x - R_x)^p = [(1)] = L_x^p - R_x^p = L_x^p - R_x^p = ad(x^p). \]
Now we get back to proving 1. Apply Claim to $A = U(g)$

$x, y \in g \subseteq U(g) \rightarrow [x^p, y] = \text{ad}(x)^p y$. Note that all operations in the right hand side in $g$. Now apply Claim to $A = \text{Mat}_n(\mathbb{F})$ (used to define $x^p$), get $[x^p, y] = \text{ad}(x)^p y$. So $[x^p - x^p, y] = \text{ad}(x)^p y - \text{ad}(x)^p y = 0 \forall y \in g$. Hence $(x) = x - x^p$ is central.

3) is similar in spirit but much harder (see Lec 10.33): for $x, y \in A$

$(x + y)^p - x^p - y^p$ is a "universal" (independent of choices of $A, x, y$) Lie polynomial (= expression in brackets) in $x, y$.

2) and 4) are easy and left as exercises. □

1.3) Completion of classification.

Let’s explain how the theorem helps in classifying the irreducible representations of $g$. Pick an irreducible $g$-representation (= $U(g)$-module) $V$. For $x \in g$, let $X_v(x)$ denote the scalar of the action of the central element $(x) \in U(g)$ on $V$, to be called the $p$-central character of $V$.

We will also need a basic tool to produce new representations from existing ones. For $g \subseteq G$, define a representation $V_g$ of $g$ as follows: if $g$ acts on $V$ via a homomorphism $p: g \rightarrow g/(V)$, then $g$ acts on $V_g$ via $p \circ \text{Ad}(g)$. By 4) of Theorem, we have (exercise):

$$X_v \circ \text{Ad}(g) = X_{vg}$$

(*)

We write $g^{*(n)}$ for the set of functions $X: g \rightarrow \mathbb{F}$ satisfying

$$X(xy) = X(x) + X(y), \quad X(ax) = a^n X(x) \quad \forall x, y \in g, a \in \mathbb{F} \quad \text{Thx to (2) & (3) of}$$
Thm, and the fact that $X_v$ is a linear function on the center we have $X_v \in g^* \forall g$-irreps $V$.

**Remark:** The group $G$ acts on $g^*$ by $g.X = X \circ \text{Ad}(g^{-1})$. Thx to (*) it's enough to classify irreps for just one $X_v$ per orbit: $V \mapsto V^*$ gives a bijection between irreps w. p-central characters $X$ and $X \circ \text{Ad}(g)$.

The next exercise describes the $G$-action on $g^*$: it's "essentially" the adjoint action on $g$:

**Exercise:**
1. Show that $(x,y) \mapsto t(xy)$ defines a $G$-invariant non-degenerate symmetric form on $g$.
2. Show that for $X \in g^*$ exist $z_x \in g$ s.t. $X(x) = \text{tr}(z_x \text{Fr}(x)) \forall x \in g$, where $\text{Fr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix})$. The map $X \mapsto z_x$ is $G$-equivariant: $X \circ \text{Ad}(g^{-1}) \mapsto \text{Fr}(g)z_x \text{Fr}(g)^{-1}$.

**Example:** $X = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These $z_x$'s are exactly representatives of all $G$-orbits (= conjugacy classes) in $g$ by the JNF theorem. This together with the remark before the exercise finishes the classification of finite dimensional irreducible representations of $g$.

2) Representations of $SL_2(\mathbb{F})$ w. char $\mathbb{F} > 2$.

Our task is to classify the irreducible rational representations. The $M(i)$'s are still there but generally are no longer irreducible.
Example: \( M(i) = \text{Span}_F(x^i, x^{i+1} y, \ldots, y^i) \) is

- for \( i < p \), irreducible over \( g \) (Sec. 3 of Lec 9) and so, since every \( G \)-subrep. is also \( g \)-subrep. over \( G \).
- \( M(p) \) is not irreducible: \( \text{Span}_F(x^p, y^p) \) is a submodule:

\[
\text{e.g. for } g = (a \ b), \text{ have } g \cdot x^p = (ax + cy)^p = a^px^p + c^py^p \subseteq \text{Span}_F(x^p, y^p).
\]

In fact, "most" of \( M(i)'s \) w. \( i < p \) are not completely reducible.

Now we produce more irreducible objects. For this we need the "Frobenius twist" construction.

Definition: Let \( V \) be a rational representation of \( G \) and \( p: G \to GL(V) \) be the corresponding homomorphism. The Frobenius twist, \( V^{(p)} \), is the representation corresponding to \( p \circ Fr: G \to GL(V) \), where, recall, \( Fr \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \left( \begin{smallmatrix} a^p & b^p \\ c^p & d^p \end{smallmatrix} \right) \), is the Frobenius homomorphism (Ex 2 in Sec 1.2 of Lec 5).

Example: \( \text{Span}_F(x^p, y^p) \cong M(i)^{(p)} \)

Observation: \( Fr \) is an abstract group isomorphism, so \( V^{(p)} \) is irreducible \( \iff \) \( V \) is.

Proposition: If \( V \) is irreducible, then \( M(i) \otimes V^{(p)} \) is irreducible \( \forall i \in \{0, \ldots, p-1\} \).

Proof: Note that \( g \) acts on \( V^{(p)} \) by \( 0 \) (\( T, Fr = 0 \)). So any \( g \)-subrepresentation of \( M(i) \otimes V^{(p)} \) is of the form \( M(i) \otimes V' \), for a subspace \( V' \subseteq V^{(p)} \).

Note that \( g(M(i) \otimes V') = M(i) \otimes gV' \). So \( M(i) \otimes V' \) is \( G \)-stable \( \iff \) \( V' \subseteq V^{(p)} \) is \( G \)-stable. Since \( V \) (hence \( V^{(p)} \)) is irreducible, we are done. \( \square \)
This gives rise to the following inductive construction. For $k \geq 0$, we write $\cdot^{(k)}$ for $\cdot^{(0)}$ repeated $k$ times.

**Corollary (Steinberg tensor product theorem)** For $0 \leq \lambda_1, \ldots, \lambda_k \leq p-1$, the representation $M(\lambda_0) \otimes M(\lambda_1)^{(0)} \otimes \cdots \otimes M(\lambda_k)^{(k)}$ is irreducible.

In fact, we'll see that these modules are pairwise non-isomorphic and exhaust all irreducible rational representations of $SL_2(F)$.

3) **Complements**: conceptual description of $x \mapsto x^{[p]}$

This part depends on the complement to Lecture 6.

- Via points of $A^*_e := \mathbb{F}(\mathfrak{g})/(\mathfrak{g}^e)$: consider the group $G(A_e)$ (see the complement to Lec 6). Let $G^+_1(A_{pt})$ be the kernel of $G(A_{pt}) \to G$ & $G^-_p(A_{pt})$ be the kernel of $G(A_{pt}) \to G(A_p)$, the latter is identified with $g_1$. Consider the map $g \mapsto g^p : G(A_{pt}) \to G(A_{pt})$.

Extending the computation in Remark of Section 1.1, we see that this map restricts to $G^+_1(A_{pt}) \to G^-_p(A_{pt})$ and moreover, factors through $g^p = G^+_1(A_{pt})/G^-_2(A_{pt}) \to G^-_p(A_{pt})$. Theorem in Section 1.1 follows.

- Via invariant vector fields: an observation is that for a commutative algebra $A$ & a derivation $S : A \to A$, the map

$S^p : A \to A$. For $A = \mathbb{F}[G]$ & left-invariant vector fields,
the map $S \mapsto S^p$ turns out to coincide with $^p$. For this one needs to prove that for $C = \text{GL}_n(F)$ we recover taking the $p$th power in $\text{gl}_n(F) = \text{Mat}_n(F)$ & then prove (2) of Thm.