Representations of algebraic groups & Lie algebras, Pt. VII

1) Weight decomposition.
2) Induced modules.
3) Completion of classification.
4) Remarks & complements.

1) Notation: $G=SL_2(\mathbb{F})$, $T=\{\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}\}\subset G$. The weight decomposition we are after comes the action of T. This is based on the following lemma. Note that $T\simeq \mathbb{F}^\times$ (via $\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}\mapsto t$).

Lemma: 1) Every rational representation of $\mathbb{F}^\times$ is completely reducible, the irreducibles are 1-dimensional,

2) and are given by $t\mapsto t^i$ ($i\in\mathbb{Z}$).

Proof: Set $S:=\{z\in \mathbb{F}^\times \mid \exists\, \ell$ coprime to p s.t. $z^\ell=1\}$ - a subgroup.

Exercise: Let V be a finite dimensional S-representation. The elements of S act by pairwise commuting diagonalizable operators. Moreover, V has an S-eigenbasis (hint: every operator from S preserves eigenspaces for any other).

Now let V be a rational representation of $\mathbb{F}^\times$. Pick an S-eigenbasis $v_1,\dots v_k$. The non-diagonal matrix coefficients vanish on S. But every polynomial function on $\mathbb{F}^\times$ (i.e. Laurent polynomial) vanishing on S is 0. So $v_1,\dots v_k$ is an eigenbasis for the entire $\mathbb{F}^\times$. This proves (1).

To prove (2) note that a rational 1-dim'l rep'n of $\mathbb{F}^\times$ is exactly a Laurent polynomial, say f, s.t. $f(st)=f(s)f(t)$. Any such is $t\mapsto t^i$ $(i\in\mathbb{Z})$. $\square$

Definition: Let V be a rational representation of G. For $i\in\mathbb{Z}$, define the i-weight space $V_i=\{v\in V:\ \begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}v=t^i v\}$.

By Lemma, $V=\bigoplus_{i\in\mathbb{Z}} V_i$.

Example: for $V=M(n)$, we have $V_{n-2i}=\mathbb{F}x^{n-i}y^i$ $(i=0,\dots n)$. The other weight spaces are zero. So the maximal (highest) weight is n, the minimal (lowest) weight is $-n$.

Exercise: 1) if V,W are rational representations of G, then

$$(V\otimes W)_i=\bigoplus_{j\in\mathbb{Z}} V_j\otimes W_{i-j}$$

2) For the Frobenius twist, $V^{(1)}$ of V (by the construction, $V^{(1)}$ & V are the same vector space) have $V_i=V^{(1)}_{pi}$ (hint: $Fr\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}=\begin{pmatrix} t^p & 0\\ 0 & t^{-p}\end{pmatrix}$).

Definition: Let $\lambda=\sum_{i=0}^{k} n_i p^i$, $n_i\in\{0,1,\dots p-1\}$. Define

$$L(\lambda):=\bigotimes_{i=0}^{k} M(n_i)^{(i)} \text{ (Frobenius twist } i \text{ times)}.$$

By Corollary in Sec 2 of Lec 10, $L(\lambda)$ is irreducible. By exercise, its highest (resp. lowest) weight is the sum of the highest (resp. lowest) weights of the factors, so is λ (resp., $-\lambda$).

The following theorem will complete the classification.

Thm: Two irreducible rational representations of $SL_2(\mathbb{F})$ w. the same lowest weight are isomorphic.

2) Induced modules.

We will see that every irreducible w. lowest weight $-\lambda$ embeds to $M(\lambda)$. For this we need to realize $M(\lambda)$ as an induced module.

Recall that if $H \subset G$ are finite groups & U is a representation of H, then the induced representation $\mathrm{Ind}_H^G U$ is defined by

$$\mathrm{Fun}_H(G,U) := \{f: G \to U \mid f(hg) = hf(g),\ \forall h \in H, g \in G\}$$

w. G-action given by $[gf](g') = f(g'g)$. We have Frobenius reciprocity:

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U) \xrightarrow{\sim} \mathrm{Hom}_H(V, U) \qquad (1)$$

Now let $H \subset G$ be algebraic groups, and U be a rational representation of H.

Definition: The (algebraic) induced representation is

$$\mathrm{Ind}_H^G U := \{\text{polynomial } f: G \to U \mid f(hg) = hf(g)\}$$

w. action of G defined as above.

Fact: For rational V, (1) holds.

The proof is standard will be given in Complements section.

Now we explain how to construct V as an induced representation. Consider the subgroup $B = \left\{\begin{pmatrix} t & u \\ 0 & t^{-1}\end{pmatrix}\right\} \subset SL_2$. Let $\mathbb{F}_{-\lambda}$ be its 1-dimensional representation where $\begin{pmatrix} t & u \\ 0 & t^{-1}\end{pmatrix}$ acts by $t^{-\lambda}$.

Proposition: $M(\lambda) = \mathrm{Ind}_B^G \mathbb{F}_{-\lambda}$.

Proof: The right hand side is $\{f \in \mathbb{F}[G] \mid f(\begin{pmatrix} t & u \\ 0 & t^{-1}\end{pmatrix} g) = t^{-\lambda} f(g)\}$

Step 1: We claim that for f satisfying

$$f(\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix} g) = f(g), \ \forall g \in SL_2, \ u \in \mathbb{F} \qquad (2)$$

$f\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is a polynomial in c & d. Indeed, since

$$\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a+uc & b+ud \\ c & d\end{pmatrix}$$

every polynomial in c & d satisfies (2).

Conversely, assume f satisfies (2). Consider the open subset $SL_2(\mathbb{F})_d$ $(d \neq 0)$ in $SL_2(\mathbb{F})$. The latter is an irreducible variety so the restriction map $\mathbb{F}[SL_2(\mathbb{F})] \to \mathbb{F}[SL_2(\mathbb{F})_d]$ is injective. The multiplication map gives rise to an isomorphism (exercise)

$$\{\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\} \times \{\begin{pmatrix} d^{-1} & 0 \\ c & d\end{pmatrix}\} \xrightarrow{\sim} SL_2(\mathbb{F})_d$$

This identifies $\mathbb{F}[SL_2(\mathbb{F})_d]$ with $\mathbb{F}[u, c, d^{\pm 1}]$ and (2) becomes "independent of u." So the elements $\underline{f} \in \mathbb{F}[SL_2(\mathbb{F})_d]$ satisfying (2) are in $\mathbb{F}[c, d^{\pm 1}]$. If $\underline{f}$ extends to SL_2, then it must be a polynomial in d, otherwise it has a pole on $\{\begin{pmatrix} 0 & c^{-1} \\ c & 0\end{pmatrix}\} = SL_2(\mathbb{F}) \setminus SL_2(\mathbb{F})_d$.

Step 2: We have $\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} ta & tb \\ t^{-1}c & t^{-1}d\end{pmatrix}$. So a polynomial in c, d lies in $\mathrm{Ind}_B^G \mathbb{F}_{-\lambda} \Leftrightarrow$ it's in $\mathrm{Span}_{\mathbb{F}}(c^\lambda, c^{\lambda-1}d, \ldots, d^\lambda)$. The claim that SL_2 acts as needed is an exercise. $\square$

3) Completion of classification.

The following proposition implies Thm in Sec 1.

Proposition: Any irreducible rational representation V of $SL_2(\mathbb{F})$ with lowest weight $-\lambda$ is isomorphic to $L(\lambda)$.

Proof: Our key claim is that
(*) $\forall$ rational representation M with $M_\mu = 0$ $\forall \mu < -\lambda$, we have $M^*_{-\lambda} \hookrightarrow \mathrm{Hom}_G(M, M(\lambda))$.

Let's explain how this implies what we want. Let M be a rational representation of $SL_2(\mathbb{F})$. By its socle, denoted soc M, we mean the maximal semisimple subrepresentation (it exists b/c the sum of two semisimple subreps is semisimple: exercise). Note that the image of any homomorphism from any irrep to M is in the socle.

By (*), $\mathrm{Hom}_G(V, M(\lambda)), \mathrm{Hom}_G(L(\lambda), M(\lambda)) \neq 0$. This means $L(\lambda), V$ are direct summands in soc $M(\lambda)$. If $V \not\cong L(\lambda)$, then

$$V \oplus L(\lambda) \hookrightarrow \mathrm{soc}\, M(\lambda) \subset M(\lambda) \Rightarrow V_{-\lambda} \oplus L(\lambda)_{-\lambda} \hookrightarrow M(\lambda)_{-\lambda}$$

Since $\dim M(\lambda)_{-\lambda} = 1$ and $\dim V_{-\lambda}, \dim L(\lambda)_{-\lambda} > 0$, this gives a contradiction.

So we need to prove (*). By Frobenius reciprocity & Proposition in Sec 2, we have, for any rational representation V,

$$\mathrm{Hom}_G(V, M(\lambda)) \xrightarrow{\sim} \mathrm{Hom}_B(V, \mathbb{F}_{-\lambda}).$$

To prove (*) we need the following claim:

(**). Let M be a rational representation of G and $m \in M_k$. Then $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} m = m_k + \sum_{i=1}^{\infty} m_{k+2i}(u)$, where m_{k+2i} is a polynomial map $\mathbb{F} \to M_{k+2i}$ (of degree i).

Proof of (**): We have $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} m = \sum_j m_j(u)$ for $m_j: \mathbb{F} \to M_j$ is a polynomial map. Observe that

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix}.$$

Apply both sides to m. Note that $\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} m = t^{-k} m$, $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} m_j(u) = t^j m_j(u)$ So $\sum_j t^{j-k} m_j(u) = \sum_j m_j(t^2 u)$. To deduce (**) from here is an exercise.

Completion of proof of (*): Set $M_{\geqslant k} = \bigoplus_{i \geqslant k} M_i$. This is a T-subrepresentation. It's stable under $\left\{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right\}$ by (**). Since T & $\left\{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right\}$ generate B, $M_{\geqslant k}$ is B-stable. By (**), on $M_{\geqslant k}/M_{\geqslant k+1}$, $\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \in B$ acts by t^k. In particular, if $M_\mu = 0$ $\forall \mu \leq -\lambda$, we have $M/M_{\geqslant 1-\lambda} \xrightarrow[B]{\sim} \mathbb{F}_{-\lambda} \otimes M_{-\lambda}$, where $M_{-\lambda}$ is the multiplicity space. So

$$M^*_{-\lambda} = \mathrm{Hom}_B(M/M_{\geqslant 1-\lambda}, \mathbb{F}_{\lambda}) \hookrightarrow \mathrm{Hom}_B(M, \mathbb{F}_{\lambda}) = \mathrm{Hom}_G(M, M(\lambda)).$$

This finishes the proof of (*) and of the proposition $\square$

Exercise: $\mathrm{soc}\, M(\lambda) = L(\lambda)$. In particular, $M(\lambda)$ is not completely reducible for $\lambda \geqslant p$.

4) Remarks & complements.

4.1) Remarks:

1) Set $W(\lambda)=M(\lambda)^*$. Its universal property is:

$$\operatorname{Hom}_G(W(\lambda),M)=\operatorname{Hom}_B(\mathbb{F}_\lambda,M)$$

Note that for $\lambda\in\mathbb{Z}$, the universal property of the Verma module $\Delta(\lambda)$ (over $\mathbb{C}$) is (Sec 1.5 of Lec 8)

$$\operatorname{Hom}_{\mathfrak{g}}(\Delta(\lambda),M)=\operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda,M),$$ where $\mathbb{C}_\lambda$ is the 1-dimensional representation of $\mathfrak{b}$, where $\begin{pmatrix}a&b\\0&-a\end{pmatrix}\in\mathfrak{b}$ acts by λa.

The two universal properties are parallel.

Another common feature: the module $W(\lambda)$ has the unique irreducible quotient (the previous exercise), the same is true for $\Delta(\lambda)$.

In fact, both $W(\lambda)$ & $\Delta(\lambda)$ are examples of "standard objects" in "highest weight categories."

2*) For those familiar w. Algebraic geometry:

The modules $M(\lambda)$ & $W(\lambda)$ are quite geometric in nature: we have $M(\lambda)=H^0(\mathbb{P}^1,\mathcal{O}(\lambda))$ - notice that $G/B=\mathbb{P}^1$ - while $W(\lambda)=H^1(\mathbb{P}^1,\mathcal{O}(-\lambda-2))$

The pairing $M(\lambda)\times W(\lambda)\to\mathbb{F}$ is $H^0(\mathbb{P}^1,\mathcal{O}(\lambda))\times H^1(\mathbb{P}^1,\mathcal{O}(-\lambda-2))\to H^1(\mathbb{P}^1,\mathcal{O}(-2))$ - Serre duality.

3) We have considered various algebraic aspects of the representation theory of $\mathfrak{sl}_2$ & SL_2. There's one we haven't considered, the most recent one, - representations of $\mathfrak{sl}_2$ in categories. This could be addressed in a future bonus lecture.

4.2) Complements.

The goal of this section is to prove Fact from Sec 2: the algebraic induction satisfies Frobenius reciprocity:

$$Hom_G(V, Ind_H^G U) \xrightarrow{\sim} Hom_H(V, U)$$

The map is as follows: let $\psi \in Hom_G(V, Ind_H^G U)$ so that $\psi(v)$ is an H-equivariant map $G \to U$, and

$$[\psi(gv)](g') = [\psi(v)](g'g)$$

We send ψ to $\alpha_\psi \in Hom(V, U)$ defined by $\alpha_\psi(v) = [\psi(v)](1)$.

Exercise: $\alpha_\psi(v)$ is H-equivariant.

Now take $\alpha \in Hom_H(V, U)$. We need to define $\psi_\alpha \in Hom_G(V, Ind_H^G U)$ In particular, $\psi_\alpha(v)$ is a map $G \to U$. Set $[\psi_\alpha(v)](g) := \alpha(gv)$. This is a polynomial map. It is H-equivariant: $[\psi_\alpha(v)](hg) = \alpha(hgv)$ $= [\alpha$ is H-equivariant$] = h\alpha(gv) = h[\psi_\alpha(v)](g)$. So ψ_α is indeed a linear map $V \to Ind_H^G(U)$. Now we check it is G-equivariant: $\psi_\alpha(gv)[g'] = \alpha(g'gv) = \psi_\alpha(v)[g'g]$.

Finally, we need to check that $\alpha \mapsto \psi_\alpha$ & $\psi \mapsto \alpha_\psi$ are inverse to each other. This is left as an exercise.