Representations of algebraic Lie groups & Lie algebras, part VIII

0) Introduction.
1) Cartan subalgebra, roots & weights.
2) $\mathfrak{sl}_2$-subalgebras, weight lattice & highest weights.
3) Verma modules and their irreducible quotients.
4) Complements.

Notation in Section 3 modified on 03/06

0) We now proceed to understanding the representation theory of simple algebraic groups & their Lie algebras. It turns out that the case of SL_n & $\mathfrak{sl}_n$ is already representative enough (outside the study representations of Lie algebras in characteristic p, where the case of $\mathfrak{sl}_n$ is significantly easier than the general case). We will concentrate on the characteristic 0 case and discuss the char p case (area of active recent & current interest) time permitting. Time permitting we will also describe generalizations: (semi) simple Lie algebras/algebraic groups and even more general Kac-Moody Lie algebras.

The three problems we are going to address for $\mathfrak{sl}_n$:

(I) The classification of finite dimensional irreducible representations.

(II) Complete reducibility of finite dimensional representations.

(III) Computation of characters of finite dimensional irreps.

We start with (I) - based on highest weight theory.

1) Cartan subalgebra, weights & roots.

Our first step in solving (I) for $\mathfrak{sl}_2$ was to decompose an $\mathfrak{sl}_2$-rep'n into the direct sum of weight spaces – the generalized eigenspaces for the element $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. This element spans the subalgebra of diagonal matrices in $\mathfrak{sl}_2$.

Now let $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of char 0.

Definition: The subalgebra of all diagonal matrices in $\mathfrak{sl}_n$: $\{diag(x_1,\dots x_n) \mid x_1+x_2+\dots+x_n=0\}$ is called a Cartan subalgebra. We denote this subalgebra by $\mathfrak{h}$.

Definition: Let V be a finite dimensional representation of $\mathfrak{g}$ and $\lambda\in\mathfrak{h}^*$. The weight space $V_\lambda := \{v\in V \mid \exists\, m>0 \mid (\xi-\langle\lambda,\xi\rangle)^m v=0,\ \forall \xi\in\mathfrak{h}\}$. We say λ is a weight of V if $V_\lambda\neq\{0\}$. A weight vector is an element of some V_λ.

Exercise: 1) if $\xi_1,\dots,\xi_{n-1}$ is a basis for $\mathfrak{h}$. Then

$$V_\lambda=\{v\in V \mid \exists\, m>0 \mid (\xi_i-\langle\lambda,\xi_i\rangle)^m v=0\}.$$

Hint: $\mathfrak{h}$ is an abelian Lie algebra, so operators in any $\mathfrak{h}$-representation pairwise commute.

2) $$V=\bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda.$$

Example: 1) Let $V=\mathbb{F}^n$ be the tautological representation of $\mathfrak{sl}_n$ w. tautological basis $e_1,\dots e_n$, weight vectors. Their weights are denoted by

$\varepsilon_1, \dots, \varepsilon_m$ so that $\varepsilon_i : \mathrm{diag}(x_1, \dots x_n) \mapsto x_i$.

2) Consider the adjoint representation, $\mathfrak{g}$. For $X = \mathrm{diag}(x_1, \dots, x_n)$ & $Y = (y_{ij}) \in \mathfrak{sl}_n$, we have $[X, Y] = ((x_i - x_j) y_{ij})$. So $\mathfrak{h} = \mathfrak{g}_0$, and for $i \neq j$, we have that $\alpha := \varepsilon_i - \varepsilon_j$ is a weight of $\mathfrak{g}$ w. $\mathfrak{g}_\alpha = \mathbb{F} E_{ij}$.

Def: Nonzero weights of $\mathfrak{g}$, i.e. $\varepsilon_i - \varepsilon_j$, are called **roots**. The **positive roots** are those w. $i<j$, equivalently, E_{ij} is upper triangular, and the **simple roots** are $\varepsilon_i - \varepsilon_{i+1}$, $i = 1, \dots, n-1$. Note that the latter form a basis of $\mathfrak{h}^*$. Note also that every positive root is a $\mathbb{Z}_{\geq 0}$-linear combination of simple roots.

Exercise: The weight spaces in $\Lambda^k \mathbb{F}^n$ are 1-dimensional, the weights are of the form $\varepsilon_{i_1} + \dots + \varepsilon_{i_k}$, $i_1 < i_2 < \dots < i_k$, and the corresponding weight vectors are $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$.

2) $\mathfrak{sl}_2$-subalgebras, weight lattice & highest weights.

Notation: for a positive root $\alpha = \varepsilon_i - \varepsilon_j$ $(i<j)$ we write e_α for E_{ij}, f_α for E_{ji} and h_α for $E_{ii} - E_{jj}$. For a simple root $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ we write e_i, f_i, h_i for $e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i}$.

Crucial observation: $e \mapsto e_\alpha, f \mapsto f_\alpha, h \mapsto h_\alpha$ defines a Lie algebra embedding $\mathfrak{sl}_2 \longrightarrow \mathfrak{g} \, (= \mathfrak{sl}_n)$. Now we can use the representation theory of $\mathfrak{sl}_2$ (Lec 8 & 9) to study that of $\mathfrak{g}$.

Lemma: 1) $\forall x \in \mathfrak{h}$, $v \in V_\lambda$ we have $xv = \langle\lambda, x\rangle v$.

2) If λ is a weight of V, then
$$\langle\lambda, h_i\rangle \in \mathbb{Z}, \ \forall i. \qquad (1)$$

3) $e_\alpha V_\lambda \subset V_{\lambda+\alpha}$, $f_\alpha V_\lambda \subset V_{\lambda-\alpha}$ $\forall$ positive roots α.

Proof: 1) It's enough to check this for every x in a basis of $\mathfrak{h}$. Both 1) for $x = h_i$ & 2) follow from i) of Proposition in Section 2 of Lec 9.

3): $xe_\alpha v = e_\alpha xv + [x, e_\alpha]v = \langle\lambda+\alpha, x\rangle v$. □

We proceed to highest weight theory, compare to Sec 1.4 in Lec 8.

Definition: • The set of $\lambda \in \mathfrak{h}^*$ satisfying (1) is called the **weight lattice**. We will denote it by Λ.

• For $\lambda, \mu \in \mathfrak{h}^*$ we write $\lambda \leq \mu$ if $\mu - \lambda$ is a $\mathbb{Z}_{\geqslant 0}$-linear combination of simple roots.

• A **highest weight** of V is a weight maximal w.r.t. this order.

• We say $\lambda \in \Lambda$ is **dominant** if $\langle\lambda, h_i\rangle \geqslant 0$ $\forall$ $i = 1, \ldots n-1$. The set of dominant weights is denoted by Λ_+.

We note that every $\lambda \in \Lambda$ can be (non-uniquely) written as $\sum_{i=1}^n \lambda_i \varepsilon_i$ w $\lambda_i \in \mathbb{Z}$, the condition that $\lambda \in \Lambda_+$ means then $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$.

Exercise: Every V has at least one highest weight. If λ is a highest weight and $v \in V_\lambda$, then $e_\alpha v = 0$ $\forall$ positive roots α.

Using (ii) of Proposition in Sec 2 of Lec 9 (for e_i, h_i, f_i) we deduce

Corollary: Every highest weight in a finite dimensional $\mathfrak{g}$-representation is dominant.

Example: In the examples from Sec 1, the highest weights are
- $\mathbb{F}^n$: $\lambda = \varepsilon_1$ w. the corresponding weight vector e_1.
- $\mathfrak{g}$: $\lambda = \varepsilon_1 - \varepsilon_n$ — — — — — — — — $e_\lambda = E_{1n}$.
- $\Lambda^k \mathbb{F}^n$: $\lambda = \varepsilon_1 + \ldots + \varepsilon_k$ — — — — — — — — $e_\lambda = e_1 \wedge e_2 \wedge \ldots \wedge e_k$.

Our goal in this and the next lecture is to prove the following result, generalizing the Lie algebra part of Thm in Sec 1.1 of Lec 8.

Thm: Every finite dimensional irreducible representation has a unique highest weight (and a unique, up to proportionality, highest weight vector). Taking the highest weight defines a bijection between the isomorphism classes of irreducibles & dominant weights.

3) Verma modules and their irreducible quotients.

We start by proving the uniqueness part of the theorem (the existence part will be proved later). As in the case of $\mathfrak{sl}_2$, we'll need the Verma modules - a universal module generated by a vector v_λ satisfying $xv_\lambda = \langle\lambda, x\rangle v_\lambda$, $e_\alpha v_\lambda = 0$, $\forall$ positive roots α.

Notation: Let $\beta_1, \ldots, \beta_N$ $(N = \frac{n(n-1)}{2})$ be the positive roots in some order. The elements $f_{\beta_j}, h_i, e_{\beta_j}$ form a basis in $\mathfrak{g}$. So the PBW theorem tells us that the elements

$$\prod_{j=1}^{N} f_{\beta_j}^{k_{\beta_j}} \prod_{i=1}^{n-1} h_i^{\ell_i} \prod_{j=1}^{N} e_{\beta_j}^{m_{\beta_j}}$$

form a basis in $\mathcal{U}(\mathfrak{g})$.

The following generalizes Definition in Sec 1.5 of Lec 8.

Definition: Let $\lambda \in \mathfrak{h}^*$. The Verma module $\Delta(\lambda)$ is $\mathcal{U}(\mathfrak{g})/I_\lambda$, where

$$I_\lambda = \mathcal{U}(\mathfrak{g})\{x - \langle\lambda, x\rangle, e_\alpha \mid x \in \mathfrak{h}, \alpha \text{ is positive root}\}$$

Set $v_\lambda := 1 + I_\lambda$.

Similarly to Proposition in Sec 1.5 of Lec 8 (and its proof) we have the following claims (exercise):

(a) $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\Delta(\lambda), V) \xrightarrow{\sim} \{v \in V \mid xv = \langle\lambda, x\rangle, e_\alpha v = 0\}$, $\forall$ $\mathfrak{g}$-representation V.

(b) The elements $\prod_{j=1}^{N} f_{\beta_j}^{k_j} v_\lambda$ form a basis in $\Delta(\lambda)$. Moreover, we

$$x \prod_{j=1}^{k} f_{\beta_j}^{k_j} v_\lambda = \langle \lambda - \sum_{j=1}^{N} k_j \beta_j, x\rangle v_\lambda, \ \forall x \in \mathfrak{h}.$$

(c) In particular,

$$\Delta(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*, \mu \leq \lambda} \Delta(\lambda)_\mu \quad \text{w. } \Delta(\lambda)_\lambda = \mathbb{F} v_\lambda.$$

(d) For any $\mathcal{U}(\mathfrak{g})$-submodule $M \subset \Delta(\lambda)$, we have $M = \bigoplus_\mu M_\mu$, $M_\mu := M \cap \Delta(\lambda)_\mu$.

When $n=2$, one can completely describe all submodules of $\Delta(\lambda)$. In general this is impossible. However, we have the following.

Proposition: $\forall \lambda \in \mathfrak{h}^*$, $\Delta(\lambda)$ has a unique <u>maximal</u> (w.r.t. $\subseteq$) submode ($\Leftrightarrow$ unique irreducible quotient, to be denoted by $L(\lambda)$).

Proof: First of all, we claim that a $\mathcal{U}(\mathfrak{g})$-submodule M is $\subsetneq \Delta(\lambda) \Leftrightarrow M_\lambda = \{0\}$. "$\Leftarrow$" follows from $\Delta(\lambda)_\lambda \neq \{0\}$; "$\Rightarrow$" follows from $\Delta(\lambda)_\lambda = \mathbb{F}v_\lambda$ & $\Delta(\lambda) = \mathcal{U}(\mathfrak{g})v_\lambda$.

Now just note that if $M^i \subset \Delta(\lambda)$ are submodules indexed by certain set I so that $M^i = [(\alpha)] = \bigoplus_\lambda M^i_\lambda$. If $M^i_\lambda = \{0\}$, $\forall i \in I$, then $(\sum_{i\in I} M^i)_\lambda = \sum_{i \in I} M_i^\lambda = \{0\}$. This finishes the proof. $\square$

The following should be compared to Sec 1.6 in Lec 8.

Corollary: Let V be an irreducible finite dimensional representation of $\mathfrak{g}$. Then $V \simeq L(\lambda)$ for a unique $\lambda \in \Lambda_+$. Moreover, $\dim V_\lambda = 1$.

Proof: V has a highest weight, Exercise in Sec 2. So $V \simeq L(\lambda)$ for some λ; $\lambda \in \Lambda_+$ (dominant weight) by Corollary in Sec 2. Note that by the construction of $L(\lambda)$, we have $\dim L(\lambda)_\lambda = \dim \Delta(\lambda)_\lambda = 1$. Also by (c) above, we have $L(\lambda)_\mu \neq \{0\} \Rightarrow \mu \leq \lambda$. This implies the uniqueness of the highest weight. $\square$

Conclusion: We have embedded the set $\mathrm{Irr}_{fd}(\mathfrak{g})$ of finite dimensional irreducible $\mathfrak{g}$-reps into the set Λ_+ of dominant weights. What remains is to prove that the image is Λ_+ - for each dominant weight there is a finite dimensional irrep. w. that highest weight $\Leftrightarrow$ for $\lambda \in \Lambda_+$, $\dim L(\lambda) < \infty$ - to be done in Lec 13.

4) Complements.

The goal of this part is to carry over the content of this lecture to the classical Lie algebras, $\mathfrak{so}_n$ & $\mathfrak{sp}_n$. We'll do the former in some detail and leave the latter as an exercise.

Recall that $\mathfrak{g}=\mathfrak{so}_n(\mathbb{F})$ can be viewed as the Lie algebra of all operators skew-symmetric w.r.t. an orthogonal (= non-degenerate symmetric) form. We take the form on $\mathbb{F}^n$ w. matrix $\begin{pmatrix} 0 & & 1\\ & \iddots & \\ 1 & & 0\end{pmatrix}$. So $\mathfrak{so}_n(\mathbb{F})$ consists of matrices skew-symmetric w.r.t. the main <u>anti-diagonal</u>.

The advantage of this choice is that now we have many diagonal matrices in $\mathfrak{so}_n$. For $n=2m$, they are of the form $\mathrm{diag}(x_1,\dots,x_m,-x_m,\dots,-x_1)$, while for $n=2m+1$, they are of the form $\mathrm{diag}(x_1,\dots,x_m,0,-x_m,\dots,-x_1)$. Let $\mathfrak{h}$ denote the subalgebra of such matrices. Let $\varepsilon_i\in\mathfrak{h}^*$ be the function sending the diagonal matrix above to x_i, $i=1,\dots,m$. The elements $\varepsilon_1,\dots,\varepsilon_m$ form a basis in $\mathfrak{h}^*$. The roots (= the nonzero weights in $\mathfrak{g}$) are as follows:

- Case $n=2m$. They are $\pm\varepsilon_i\pm\varepsilon_j$ w $1\leqslant i<j\leqslant m$. The corresponding weight spaces in $\mathfrak{g}$ are 1-dimensional w. basis vectors
  - $E_{i,j}-E_{n+1-j,n+1-i}$ for $\alpha=\varepsilon_i-\varepsilon_j$ $(i\neq j)$: $\begin{pmatrix} & 1 & & \\ & & & \\ & & & \\ & & -1 & \end{pmatrix}$
  - $E_{i,n+1-j}-E_{j,n+1-i}$ for $\alpha=\varepsilon_i+\varepsilon_j$ $(i<j)$: $\begin{pmatrix} & & 1 & \\ & & & -1\\ & & & \\ & & & \end{pmatrix}$
  - $E_{n+1-i,j}-E_{n+1-j,i}$ for $\alpha=-\varepsilon_i-\varepsilon_j$ $(i<j)$: $\begin{pmatrix} & & & \\ & & & \\ 1 & & & \\ & -1 & & \end{pmatrix}$

- Case: $n=2m+1$: The roots are $\pm\varepsilon_i\pm\varepsilon_j$ w. $1\leqslant i<j\leqslant m$ and also $\pm\varepsilon_i$ w. $1\leqslant i\leqslant m$.

We say that a root is positive if it's $\varepsilon_i \pm \varepsilon_j$ w $i<j$ (for $n=2m$) or $\varepsilon_i \pm \varepsilon_j$ $(i<j)$ and ε_i (for $n=2m+1$), equivalently the corresponding root vector is an upper triangular matrix. This uniquely specifies the simple roots – a minimal collection of positive roots s.t every positive root is their $\mathbb{Z}_{\geqslant 0}$-linear combination. The simple roots are as follows:

$n=2m$: $\alpha_1=\varepsilon_1-\varepsilon_2, \alpha_2=\varepsilon_2-\varepsilon_3, \ldots, \alpha_{m-1}=\varepsilon_{m-1}-\varepsilon_m, \alpha_m=\varepsilon_{m-1}+\varepsilon_m.$

$n=2m+1$: $\alpha_1=\varepsilon_1-\varepsilon_2, \ldots, \quad \alpha_{m-1}=\varepsilon_{m-1}-\varepsilon_m, \alpha_m=\varepsilon_m.$

Now pick a positive root α. We can normalize the vectors e_α (of weight α) and f_α (of weight $-\alpha$) so that e_α, f_α, and $h_\alpha=[e_\alpha, f_\alpha]$ satisfy the $\mathfrak{sl}_2$-relations: $[h_\alpha, e_\alpha]=2e_\alpha, [h_\alpha, f_\alpha]=-2f_\alpha, [e_\alpha, f_\alpha]=h_\alpha$. The vector h_α is determined uniquely. For the simple roots α_i the vectors h_i are as follows

$n=2m$: $h_i=\operatorname{diag}(0,..0,\underset{\text{ith slot}}{1},-1,0,..0,-1,1,..0), i<m, \; h_m=(0,..,0,1,1,-1,-1,0,..0)$

$n=2m+1$: h_i – similar, if $i<m$, $h_m=(0,..0,2,0,-2,0,..0)$.

With this the representation-theoretic stuff in Sections 1-3 goes through: we have the finite dimensional irreducible representations classified by dominant weights (and so far the conclusion of Section 3 is reached).

Exercise: work out the $\mathfrak{sp}_n$ (n is even) case.