Representations of algebraic groups & their Lie algebras, IX.

1) $\dim L(\lambda) < \infty$ for $\lambda \in \Lambda_+$

2) Complements.

1.0) Recap.

Recall that $\mathbb{F}$ is an algebraically closed field of char 0, $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{F})$, $\mathfrak{h} = \{diag(x_1,...,x_n) \mid x_1+...+x_n = 0\}$, the Cartan subalgebra.

For $i = 1,...,n-1$, we consider the elements $h_i = E_{ii} - E_{i+1,i+1}$, a basis in $\mathfrak{h}$. Inside $\mathfrak{h}^*$ we consider subsets $\Lambda \supset \Lambda_+$ consisting of all "weights", $\lambda \in \mathfrak{h}^* \mid \langle\lambda, h_i\rangle \in \mathbb{Z}, \forall i$, and all dominant weights: $\lambda \in \Lambda$ w. $\langle\lambda, h_i\rangle \geqslant 0$.

Explicitly, $\Lambda = \{\sum_{i=1}^n \lambda_i \varepsilon_i \mid \lambda_i \in \mathbb{Z}\} \supset \Lambda_+ = \{\sum_{i=1}^n \lambda_i \varepsilon_i \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n\}$

We have a partial order on $\mathfrak{h}^*$: $\lambda \leqslant \mu \Leftrightarrow \mu - \lambda \in Span_{\mathbb{Z}_{\geqslant 0}}(\varepsilon_i - \varepsilon_j \mid i<j)$

Explicitly, on Λ this means:

$$\lambda \leqslant \mu \Leftrightarrow \sum_{i=1}^k \lambda_i \leqslant \sum_{i=1}^k \mu_i \ \forall k<n \ \& \ \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i \text{ (exercise)}.$$

Let $\mathfrak{n} \subset \mathfrak{g}$ be the subalgebra of strictly upper triangular matrices: $\mathfrak{n} := Span_{\mathbb{F}}(E_{ij} \mid i<j) = \{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}\}$.

For $\lambda \in \mathfrak{h}^*$ we have defined the Verma module

$$\Delta(\lambda) := \mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) Span_{\mathbb{F}}\{x - \langle\lambda, x\rangle, y \mid x \in \mathfrak{h}, y \in \mathfrak{n}\} \ni v_\lambda := 1 + I_\lambda.$$

We've seen, Proposition in Sec 3 of Lec 12, that $\Delta(\lambda)$ has the unique irreducible quotient, $L(\lambda)$. By Corollary in Sec 3 of Lec 12, $\forall$ finite dimensional irrep V of $\mathfrak{g}$ $\exists! \lambda \in \Lambda_+$ w. $V \simeq L(\lambda)$.

Goal for today: $\forall \lambda \in \Lambda_+ \Rightarrow \dim L(\lambda) < \infty$.

This will finish the classification of finite dimensional $\mathfrak{g}$-irreps.

1.1) Explicit constructions.

Define the fundamental weights $\omega_i \in \mathfrak{h}^*$, $i=1,\dots,n-1$, $\omega_i := \sum_{j=1}^{i} \varepsilon_j$ so that $\langle \omega_i, h_j\rangle = \delta_{ij}$. Every $\lambda \in \Lambda_+$ is uniquely written as $\sum_{i=1}^{n-1} n_i \omega_i$ w. $n_i = \langle \lambda, h_i\rangle \in \mathbb{Z}_{\geq 0}$.

Example: Now we discuss $L(\omega_k)$. We claim it's isomorphic to $\Lambda^k \mathbb{F}^n$ (w. $\mathfrak{g}$-action given by $\xi(v_1\wedge\dots\wedge v_k) = \sum_{i=1}^{k} v_1\wedge\dots\wedge\xi v_i\wedge\dots\wedge v_k$, $\forall \xi\in\mathfrak{g}$. Indeed, ω_k is a highest weight of $\Lambda^k\mathbb{F}^n$ (Ex in Sec 2 of Lec 12). Moreover, $\Lambda^k\mathbb{F}^n$ is irreducible. The weights are $\lambda = \varepsilon_{i_1}+\dots+\varepsilon_{i_k}$ & $(\Lambda^k\mathbb{F}^n)_\lambda = \mathbb{F} e_{i_1}\wedge\dots\wedge e_{i_k}$ & then we use $E_{j_\ell i_\ell}(e_{i_1}\wedge\dots\wedge e_{i_\ell}\wedge\dots\wedge e_{i_k}) = e_{i_1}\wedge\dots\wedge e_{j_\ell}\wedge\dots\wedge e_{i_k}$ to see that if a subrepresentation contains $e_{i_1}\wedge\dots\wedge e_{i_k}$, it contains all $e_{j_1}\wedge\dots\wedge e_{j_k}$. So $\Lambda^k\mathbb{F}^n$ is irreducible w. highest weight ω_k. So by Cor in Sec 3 of Lec 12, $V\simeq L(\omega_k)$.

Proposition: $\dim L(\lambda) < \infty$ $\forall \lambda\in\Lambda_+$.

Proof: First, let V^1, V^2 be finite dimensional $\mathfrak{g}$-reps w. highest weights λ_1, λ_2 & $v_i \in V^i_{\lambda_i}\setminus\{0\}$. Then $v_1\otimes v_2 \in (V^1\otimes V^2)_{\lambda_1+\lambda_2}$ (from $x(v_1\otimes v_2) = xv_1\otimes v_2 + v_1\otimes xv_2$, compare to Exercise 1 in Sec 1 of Lec 11). Thx to above, $\forall \lambda\in\Lambda_+$ can find an iterated $\otimes$-product $\mathcal{U}$ of $\Lambda^k\mathbb{F}^n$'s w. highest weight λ. Take nonzero $u\in\mathcal{U}_\lambda$. $\exists!$ $U(\mathfrak{g})$-module homomorphism $\Delta(\lambda)\xrightarrow{\varphi}\mathcal{U}$, $v_\lambda\mapsto u$. Then $\Delta(\lambda)\twoheadrightarrow \operatorname{im}\varphi = U(\mathfrak{g})u$. Since $L(\lambda)$ is the unique irreducible quotient of $\Delta(\lambda)$, $U(\mathfrak{g})u \twoheadrightarrow L(\lambda)$. $\square$

Remarks: 1) We'll see later that $U(\mathfrak{g})u$ is irreducible.

2) The representation $\mathcal{U}$ in the proof comes from a rational G-rep, $G = SL_n(\mathbb{F})$. Since every $\mathfrak{g}$-stable subspace is also G-stable one can

see that $L(\lambda)$ also comes from a rational G-representation. Thx to Thm 2 in Sec 1.3 of Lec 6, each rational irreps is isomorphic to exactly one of irreps giving $L(\lambda)$.

Example: Consider $\lambda = d\omega_1 = d\varepsilon_1$. We take $U = (\mathbb{F}^n)^{\otimes d}$, $u = e_1^{\otimes d}$. Then $u \in S^d(\mathbb{F}^n)$, a $\mathfrak{g}$-subrepresentation. It's irreducible (exercise). Note that for $d=2$, recover $M(d) = S^d(\mathbb{F}^2)$).

Remark: The two examples can be generalized to arbitrary λ - via Schur-Weyl duality, Sec 5.18 in [E]. Namely observe that $(\mathbb{F}^n)^{\otimes d}$ is a representation of $GL_n(\mathbb{F}) \times S_d$, where S_d acts by permuting tensor factors. Pick a partition $\lambda = (\lambda_1, \dots \lambda_n)$ of d and form the corresponding S_d-irrep V_λ, Section 5.2 in [RT1]. Then $\operatorname{Hom}_{S_d}(V_\lambda, (\mathbb{F}^n)^{\otimes d})$ is a GL_n-irrep. Moreover, the corresponding representation of $\mathfrak{sl}_n$ is $L(\sum_{i=1}^n \lambda_i \varepsilon_i)$. E.g. $S^d(\mathbb{F}^n)$ & $\Lambda^d(\mathbb{F}^n)$ arise from $V_{(d)} = triv_d$, $V_{(1^d)} = sgn_d$, respectively.

1.2) Homomorphisms between Verma modules.

Now our goal is to produce a finite dimensional quotient $\tilde{L}(\lambda)$ of $\Delta(\lambda)$ $(\lambda \in \Lambda_+)$ w. explicit relations. This will show $\dim L(\lambda) < \infty$. This is much more involved than what we did in the previous section but uses several important constructions, incl. one in the title, and reveals important structure. Our first step is to establish a homomorphism between some Verma modules.

Pick $i = 1, \dots, n-1$, $\lambda \in \Lambda$ w. $\langle \lambda, h_i \rangle \geq 0$. Set $m = \langle \lambda, h_i \rangle + 1$, $\lambda_i' = \lambda - m\alpha_i$.

Proposition: $\dim \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\Delta(\lambda_i'),\Delta(\lambda))=1$.

Proof: Let $\beta_1,\dots,\beta_N$ be all positive roots. For $\alpha=\varepsilon_i-\varepsilon_j$ $(i<j)$, we set $f_\alpha=E_{ji}\in\mathfrak{g}_{-\alpha}$. Then, by (6) in Section 3 of Lec 12, the vectors $\prod_{k=1}^N f_{\beta_k}^{m_k} v_\lambda$ form a basis in $\Delta(\lambda)$ and the weight of this vector is $\lambda-\sum_{k=1}^N n_k\beta_k$.

Some notation: we write e_i for $E_{i,i+1}$, f_i for f_{α_i}, $\mathfrak{s}_i$ for $\operatorname{Span}_{\mathbb{F}}(e_i,h_i,f_i)$, it's a subalgebra of $\mathfrak{g}$ w. $\mathfrak{sl}_2\xrightarrow{\sim}\mathfrak{s}_i$.

Recall that by the universal property of $\Delta(\lambda_i')$ ((a) in Sec 3 of Lec 12) we have

$$\operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\Delta(\lambda_i'),\Delta(\lambda))=\{v\in\Delta(\lambda)_{\lambda_i'}\mid \mathfrak{n}v=0\}$$

Exercise: $\Delta(\lambda)_{\lambda_i'}=\mathbb{F}(f_i^m v_\lambda)$ (hint: analyze the condition $\lambda-\sum_{k=1}^N n_k\beta_k=\lambda-m\alpha_i$).

It remains to show $\mathfrak{n}f_i^m v_\lambda=0$. This follows from

Exercise: • the Lie algebra $\mathfrak{n}$ is generated by the elements e_j, $j=1,\dots,n-1$.

• We have $e_jf_i^m v_\lambda=0$ $\forall j$ (hint: $[e_j,f_i]=0$ for $j\neq i$, while the case of $j=i$ is an $\mathfrak{sl}_2$-computation done in the proof of 3) of Proposition in Sec 1.5 of Lec 8): $e_if_i^m=f_i^me_i+mf_i^{m-1}(h_i-(m-1))$. $\square$

1.3) Finite dimensional quotient of $\Delta(\lambda)$.

Suppose $\lambda \in \Lambda^+$; $\lambda_i' = \lambda - (\langle\lambda, h_i\rangle + 1)\alpha_i$.
Let $\varphi_i : \Delta(\lambda_i') \longrightarrow \Delta(\lambda)$ be a nonzero homomorphism (it's actually injective but we don't need this). Suppose $\lambda \in \Lambda^+$. Set

$$\tilde{L}(\lambda) = \Delta(\lambda) / \sum_{i=1}^{n-1} \underbrace{\operatorname{im} \varphi_i}$$

independent of the choice of φ_i, Prop in Sec 1.2.

Proposition: $\dim \tilde{L}(\lambda) < \infty$.

Note that $\tilde{L}(\lambda) \twoheadrightarrow L(\lambda)$. So we see that $\dim L(\lambda) < \infty$.
Later on we'll see that $\tilde{L}(\lambda) = L(\lambda)$.

We introduce some more notation. To a simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ we assign $s_i \in GL(\mathfrak{h}^*)$, $s_i \lambda = \lambda - \langle\lambda, h_i\rangle\alpha_i$. More explicitly,

$$s_i \sum_{i=1}^{n} n_i \varepsilon_i = \sum_{i=1}^{n} n_i \varepsilon_i - (n_i - n_{i+1})(\varepsilon_i - \varepsilon_{i+1}) = n_1\varepsilon_1 + \ldots + n_{i+1}\varepsilon_i + n_i\varepsilon_{i+1} + \ldots + n_n\varepsilon_n.$$

Definition: The subgroup of $GL(\mathfrak{h}^*)$ generated by s_i is called the Weyl group of G & is denoted by W.

Exercise: W is just S_n acting in its reflection representation, $\mathfrak{h}^*$.

Proof of Prop'n: The outline of the proof is as follows:

i) We show that every vector in $\tilde{L}(\lambda)$ is contained in a finite dimensional $\mathfrak{s}_i$-subrepresentation.

ii) We observe that $\tilde{L}(\lambda)_\mu \neq 0 \Rightarrow \mu \leq \lambda$ & $\dim \tilde{L}(\lambda)_\mu < \infty$

iii) We show that (i) & (ii) imply an isomorphism $\tilde{Z}(\lambda)_\mu \xrightarrow{\sim} \tilde{Z}(\lambda)_{s_i\mu}$ $\forall \mu \in \Lambda$.

iv) We deduce the claim that $\dim \tilde{Z}(\lambda) < \infty$ from ii) & iii).

i) Let $\underline{v}_\lambda$ denote the image of $v_\lambda \in \Delta(\lambda)$ in $\tilde{Z}(\lambda)$. Since $\mathcal{U}(\mathfrak{g})v_\lambda = \Delta(\lambda)$, we see that $\mathcal{U}(\mathfrak{g})\underline{v}_\lambda = \tilde{Z}(\lambda)$. So, it's enough to show that $a\underline{v}_\lambda$ lies in a finite dimensional $\mathfrak{s}_i$-subrepresentation, $\forall a \in \mathcal{U}(\mathfrak{g})$.

Note that $\mathcal{U}(\mathfrak{g})$ can be viewed as a representation of $\mathfrak{g}$ via ad: $\mathrm{ad}(\xi)a = [\xi, a]$. We claim that a lies in a finite dimensional $\mathrm{ad}(\mathfrak{g})$-stable subspace, $U_0 \subset \mathcal{U}(\mathfrak{g})$. Note that $\exists i > 0 \mid a \in \mathcal{U}(\mathfrak{g})_{\leq i} = \mathrm{Span}(\xi_1\xi_2\cdots\xi_k \mid \xi_1, \dots \xi_k \in \mathfrak{g}, k \leq i)$.

Exercise: $\mathcal{U}(\mathfrak{g})_{\leq i}$ is finite dimensional and $\mathrm{ad}(\mathfrak{g})$-stable (hint for the latter: $[\xi, \xi_1 \cdots \xi_k] = [\xi, \xi_1]\xi_2 \cdots \xi_k + \xi_1[\xi, \xi_2]\xi_3 \cdots \xi_k + \dots + \xi_1 \cdots \xi_{k-1}[\xi, \xi_k]$).

So, set $U_0 := \mathcal{U}(\mathfrak{g})_{\leq i}$. And since $f_i^m \underline{v}_\lambda = 0$, we see that $V_0 := \mathrm{Span}_{\mathbb{F}}(f_i^j \underline{v}_\lambda \mid j = 0, \dots, m-1)$ is $\mathfrak{s}_i$-stable (and finite dimensional)

Now we proceed to producing a finite dimensional $\mathfrak{s}_i$-stable subspace of $\tilde{Z}(\lambda)$ containing $a\underline{v}_\lambda$.

Exercise: For every $\mathfrak{g}$-representation V, the action map $\alpha: \mathcal{U}(\mathfrak{g}) \otimes V \to V$, $a \otimes v \mapsto av$, is $\mathfrak{g}$-linear.

In particular, $\alpha: \mathcal{U}(\mathfrak{g}) \otimes \tilde{Z}(\lambda) \to \tilde{Z}(\lambda)$ is $\mathfrak{g}$- & hence $\mathfrak{s}_i$-linear. So $a\underline{v}_\lambda = \alpha(a \otimes \underline{v}_\lambda) \subset \alpha(U_0 \otimes V_0)$, a finite dimensional $\mathfrak{s}_i$-stable subspace. This finishes (i).

ii) Since $\Delta(\lambda) \twoheadrightarrow \tilde{L}(\lambda)$, it's enough to establish these properties for $\Delta(\lambda)$ instead of $\tilde{L}(\lambda)$. Recall that the weight vectors $\prod_{j=1}^{N} f_{\beta_j}^{k_j} v_\lambda$ of weight $\lambda - \sum_{j=1}^{N} k_j \beta_j$ form a basis in $\Delta(\lambda)$. So

$$\dim \Delta(\lambda)_\mu = \#\{(k_1, \ldots, k_N) \in \mathbb{Z}_{\geq 0}^N \mid \lambda - \mu = \sum_{j=1}^{N} k_j \beta_j\}.$$

Exercise: the set in the r.h.s. is finite (and nonzero $\Leftrightarrow \mu \leq \lambda$).

iii) We can assume $s_i \mu \neq \mu$. By (ii), $\tilde{L}(\lambda)_\mu \oplus \tilde{L}(\lambda)_{s_i \mu}$ is finite dimensional. By i), $\exists$ finite dimensional $\mathfrak{s}_i$-subrepresentation $V \subset \tilde{L}(\lambda)$ that contains $\tilde{L}(\lambda)_\mu \oplus \tilde{L}(\lambda)_{s_i\mu}$. Note that h_i acts on $\tilde{L}(\lambda)_\mu$ by $\langle \mu, h_i \rangle =: \ell$, and on $\tilde{L}(\lambda)_{s_i\mu}$ by $\langle \mu - \langle \mu, h_i \rangle \alpha_i, h_i \rangle = \langle \mu, h_i \rangle - \langle \mu, h_i \rangle \langle \alpha_i, h_i \rangle =$ $= \langle \mu, h_i \rangle - 2\langle \mu, h_i \rangle = -\langle \mu, h_i \rangle = -\ell$. So $\tilde{L}(\lambda)_\mu \subset V_\ell$, $\tilde{L}(\lambda)_{s_i\mu} \subset V_{-\ell}$, where $V_\ell = \{v \in V \mid hv = \ell v\}$ & $V_{-\ell} = \{v \in V \mid hv = -\ell v\}$. Assume $\ell \geq 0$ w.l.o.g. Recall, (iii) of Proposition in Sec 2 of Lec 9, that $e_i^\ell : V_{-\ell} \xrightarrow{\sim} V_\ell$, $f_i^\ell : V_\ell \to V_{-\ell}$. Also $e_i^\ell \tilde{L}(\lambda)_{s_i\mu} \subset \tilde{L}(\lambda)_\mu$, $f_i^\ell \tilde{L}(\lambda)_\mu \subset \tilde{L}(\lambda)_{s_i\mu}$. Since $\dim V_\ell = \dim V_{-\ell} < \infty$, we conclude that $\dim \tilde{L}(\lambda)_\mu = \dim \tilde{L}(\lambda)_{s_i\mu}$.

iv) Since all weight spaces in $\tilde{L}(\lambda)$ are finite dimensional, (ii), it's enough to show that there are only finitely many weights of $\tilde{L}(\lambda)$. We'll deduce this from (iii). Consider the element $w_0 \in S_n$ given by $w_0(i) := n+1-i$. It sends all positive roots to negative roots, so reverses $\leq$. From (iii), the set of weights of $\tilde{L}(\lambda)$ is S_n-stable. So if μ is a weight, then so is $w_0\mu$. From (ii), $\mu, w_0\mu \leq \lambda \Leftrightarrow [w_0^2 = id]$ $w_0\lambda \leq \mu \leq \lambda$. The set of μ satisfying these inequalities is finite $\square$

3) Complements.

Our goal here is to see how the construction of the irreducible representations corresponding to the fundamental weights carries over to the orthogonal and symplectic Lie algebras. For $\mathfrak{sl}_n$, this is Sec 1.1.

We use the notation of the complement section of Lec 12.

3.1) Symplectic case: this case is quite similar to the case of $\mathfrak{sl}_n$. The fundamental weights are $\varepsilon_1, \varepsilon_1+\varepsilon_2, \dots, \varepsilon_1+\varepsilon_2+\dots+\varepsilon_m$ $(m=n/2)$. Let V be the tautological representation of $\mathfrak{sp}_n$, with weights $\varepsilon_1, \dots \varepsilon_m, -\varepsilon_1, \dots -\varepsilon_m$. The highest weight of $\Lambda^k V$ is $\varepsilon_1+\dots+\varepsilon_k$. With some (multilinear algebra) work one can prove that $\Lambda^k V \simeq L(\omega_k)\oplus L(\omega_{k-2})\oplus \dots \oplus L(\omega_{k-2\lfloor k/2\rfloor})$.

3.2) Orthogonal case: this case establishes new features. Their origin is that while $SL_n(\mathbb{C})$, $Sp_n(\mathbb{C})$ are simply connected, the group $SO_n(\mathbb{C})$ is not: the fundamental group is $\mathbb{Z}/2\mathbb{Z}$. So one should expect that there are irreducible representations of $\mathfrak{so}_n(\mathbb{C})$ that do not come from rational representations of $SO_n(\mathbb{C})$. This is indeed the case – (half) spinor representations. They have to do with modules over the Clifford algebras.

Definition: Let V be a finite dimensional vector space over $\mathbb{F}$ w. an orthogonal form B. By the Clifford algebra $Cl(V,B)(=Cl(V))$ one means the quotient

$$T(V)/(u\otimes v+v\otimes u-B(u,v)\,|\,u,v\in V).$$

Exercise: • Let $v_1, \dots v_n$ be an orthogonal basis of V. Then the elements $v_{i_1} v_{i_2} \dots v_{i_k}$ w. $i_1 < i_2 < \dots < i_k$ form a basis of $Cl(V,B)$. In particular, $\dim Cl(V,B) = 2^n$.

• The span of the basis elements of the form $v_{i_1} \dots v_{i_k}$ w. even k is a subalgebra denoted by $Cl^+(V)$.

Let's now explain of connection between $\mathfrak{so}(V)$ & $Cl^+(V)$.

Exercise: In the notation of the previous proposition, $\mathrm{Span}_{\mathbb{F}}(v_i v_j - v_j v_i) \subset Cl^+(V)$ is a Lie subalgebra isomorphic to $\mathfrak{so}(V)$. Moreover, it generates $Cl^+(V)$.

It follows that restricting an irreducible representation from $Cl^+(V)$ to $\mathfrak{so}(V)$ we get an irreducible representation. It turns out that $Cl^+(V)$ is isomorphic to the matrix algebra of size $2^{(n-1)/2}$ if n is odd, and to the direct sum of two matrix algebras of size $2^{(n-2)/2}$ when n is even.

First, suppose $\dim V = 2m$ is even and consider $Cl(V)$. Pick a Lagrangian subspace $L \subset V$, i.e. a subspace of dim m s.t. the restriction of B to L is zero. Then a complement to L in V is identified w. L^* via $v \in V \mapsto B(v, \cdot): L \to \mathbb{F}$. So we can decompose V as $L \oplus L^*$.

Let's construct a $Cl(V)$-module structure on the exterior algebra $\Lambda L := \bigoplus_{i=0}^{m} \Lambda^i m$. Here an element $\ell \in L \subset V$ acts on ΛL by the multiplication by ℓ. An element $\alpha \in L^*$ sends the monomial $\ell_1 \wedge \ell_2 \wedge \dots \wedge \ell_k$ to $\sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, \ell_i \rangle \ell_1 \wedge \dots \wedge \ell_{i-1} \wedge \ell_{i+1} \wedge \dots \ell_k$.

Exercise: Check that this extends to a representation of $Cl(V)$ in ΛL. Moreover, this representation is irreducible.

Comparing the dimensions, we see that $Cl(V) \xrightarrow{\sim} End(\Lambda L)$.

Now let's proceed to $Cl^+(V)$, still for V of even dimension. It's an algebra of dimension 2^{2m-1}. We can decompose ΛL as $\Lambda^{even} L \oplus \Lambda^{odd} L$, where $\Lambda^{even} L$ is the sum of even exterior powers & $\Lambda^{odd} L$ is defined similarly. One easily sees that these two subspaces are $Cl^+(V)$-stable. So the isomorphism $Cl(V) \xrightarrow{\sim} End(\Lambda L)$ restricts to

$$Cl^+(V) \hookrightarrow End(\Lambda^{odd} L) \oplus End(\Lambda^{even} L)$$

which, for dimension reasons, is an isomorphism. Our conclusion is that $\Lambda^{odd} L$, $\Lambda^{even} L$ are irreducible $Cl^+(V)$-modules, hence irreducible $\mathfrak{so}(V)$-modules (for $\dim V$ even).

To understand the weights we need to understand the elements of $Cl^+(V)$ corresponding to diagonal matrices in $\mathfrak{so}_{2m}$. Let $e_1, \dots, e_{2m}$ be the tautological basis for $\mathfrak{so}_{2m}$ (= matrices skew-symmetric w.r.t. the main anti-diagonal). Then the diagonal matrix $E_{ii} - E_{n+1-i, n+1-i}$ $(n=2m)$ corresponds to $\frac{1}{2}(e_i e_{n+1-i} - e_{n+1-i} e_i)$, exercise.

Pick $L = \mathrm{Span}(e_1, \dots e_m)$. Then $\frac{1}{2}(e_i e_{n+1-i} - e_{n+1-i} e_i)$ acts on $1 \in \Lambda L$ by $-\frac{1}{2}$.

So 1 has weight $-\frac{1}{2}(\varepsilon_1+\ldots+\varepsilon_m)$. And $e_{i_1}\wedge\ldots\wedge e_{i_k}\in\Lambda L$ has weight $-\frac{1}{2}(\varepsilon_1+\ldots+\varepsilon_m)+\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}$. So the highest weights of $\Lambda^{odd}L$ & $\Lambda^{even}L$ are $\frac{1}{2}(\varepsilon_1+\ldots+\varepsilon_m)=\omega_m$ & $\frac{1}{2}(\varepsilon_1+\ldots+\varepsilon_{m-1}-\varepsilon_m)=\omega_{m-1}$. The corresponding $\mathfrak{so}_{2m}$-irreps are the half-spinor representations.

To handle the case when n is odd, $n=2m+1$, we make the following observation. Let $\tilde{V}$ be an orthogonal vector space of dimension $2m+1$. Pick a vector $v_0\in\tilde{V}$ w. $(v_0,v_0)=-\frac{1}{2}$. Let $V=v_0^\perp$. The map $V\longrightarrow Cl^+(\tilde{V})$, $v\mapsto v_0v$, extends to an algebra homomomorphism $Cl(V)\longrightarrow Cl^+(\tilde{V})$, that is an isomorphism (e.g. it's not hard to see that $\mathfrak{so}(\tilde{V})$ is in the image). So $Cl^+(\tilde{V})$ has the unique irreducible representation. One can compute that its weights for $\mathfrak{so}(\tilde{V})$ are $\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\ldots\pm\varepsilon_m)$. So the highest weight is $\omega_m=\frac{1}{2}(\varepsilon_1+\ldots+\varepsilon_m)$. We get the spinor representation.

The two irreducible representations of $\mathfrak{so}_{2m}$ and the irreducible representation of $\mathfrak{so}_{2m+1}$ come not from representations of SO, but of its simply connected 2-fold cover, the Spin group.

The rest of irreducible representations whose highest weight is fundamental (ω_k w. $k<m-1$ for $n=2m$ & $k<m$ for $n=2m+1$) are easy - those are $\Lambda^k\mathbb{F}_n$ (exercise).