Representations of $\mathfrak{sl}_2$ in categories, part 1.

0) Introduction.

1) Categories & functors.

0) Much of relatively modern (last 50 years or so) representation theory deals with understanding categories of representations. And a general paradigm to understand all kinds of objects (and forming a foundation of Representation theory, in particular) is to understand their symmetry. In the context of studying representations of semisimple Lie algebras, this idea was very essentially used by Soergel in the 90's, more on this later in the class. Also in the 90's it was envisioned by Igor Frenkel that a categorical symmetry that is a counterpart of representations of semisimple Lie algebras (of which $\mathfrak{sl}_2$ is the simplest example) should play an important role (in particular, for studying invariants of knots & links - that has been realized since then - and of 3-manifolds - that hasn't). A bit later, it was discovered, e.g. by Chuang and Rouquier that this theory should be useful for Representation theory, including modular representations of symmetric groups. We'll discuss some things about the categorical $\mathfrak{sl}_2$-symmetry in this note and subsequent ones.

Rem: This is not a serious remark. In Hegel's philosophy there's the notion of a "triad": thesis $\rightarrow$ antithesis $\rightarrow$ synthesis. In our situation these could be as follows:

Thesis: we want to understand concrete representations (of groups, algebras etc.)

Antithesis: concrete realizations are not important, the structure of a category is.

Synthesis: we'll understand categories by studying their concrete symmetries.

References: Kleshchev's book & Chuang, Rouquier, "Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification"

1) Categories & functors.

We are going to explain the most basic setup. Let $\mathbb{F}$ be a field. Let $\mathcal{C}$ be a category equivalent to a direct sum of categories of finite dimensional representations of finite dimensional associative algebras. A good (and relevant) example is $\mathcal{C} = \bigoplus_{n\geqslant 0} \mathbb{F}S_n\text{-mod}$. Every object is a formal finite direct sum of representations of different S_n's. The space of morphisms from $\bigoplus_i M_i$ to $\bigoplus_i N_i$ ($M_i, N_i \in \mathbb{F}S_i\text{-mod}$) is $\bigoplus_i \operatorname{Hom}_{S_i}(M_i, N_i)$.

Rem: More generally, we need $\mathcal{C}$ to be an $\mathbb{F}$-linear abelian category, where every object has finite length, i.e. admits a finite JH filtration.

1.1) K_0.

From a category $\mathcal{C}$ we can form a free abelian group, its

Grothendieck group, denoted by $K_0(\mathcal{C})$. Namely, it makes sense to speak about simple objects in $\mathcal{C}$, those w/o proper subobjects. For example, in $\bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod, the simple objects are exactly the irreducible representations of all symmetric groups.

By definition, $K_0(\mathcal{C})$ is the free abelian group w. basis $[L]$, where L runs over the set of isomorphism classes of simple objects in $\mathcal{C}$. For an arbitrary object, $M \in Ob(\mathcal{C})$, we define $[M] \in K_0(\mathcal{C})$ as $[M] = \sum_{i=1}^{k} [M_i / M_{i-1}]$, where $\{0\} = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k$ is a JH filtration. It's well-defined by the JH theorem.

Now suppose that $\mathcal{C}$ & $\mathcal{D}$ are two categories of the type we consider. Suppose $F: \mathcal{C} \longrightarrow \mathcal{D}$ is an exact functor. Then we have a group homomorphism $[F]: K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D})$, $[M] \longmapsto [F(M)]$, details are an exercise.

1.2) Categorical $\mathfrak{sl}_2$-action, the first attempt.

Now that we can convert categories (of special kind) to abelian groups (and hence, by $\mathbb{C} \otimes_{\mathbb{Z}} \bullet$, to complex vector spaces). We can also convert exact functors to linear maps. So, perhaps, we could try to define an action of $\mathfrak{sl}_2$ on $\mathcal{C}$ as a triple of exact functors $E, H, F: \mathcal{C} \rightarrow \mathcal{C}$ s.t. the operators $[E], [H], [F]$ on $K_0(\mathcal{C})$ satisfying the defining relations of $\mathfrak{sl}_2$. However, this is not what's observed in examples – while the operators e, f do come from functors, h doesn't. But if we consider "weight modules", then we don't need h as an operator.

Definition: A weight representation of $\mathfrak{sl}_2'$ is a vector space V together w. a direct sum decomposition $V=\bigoplus_{i\in\mathbb{Z}} V_i$ and two linear operators e,f s.t.

(c1) $eV_i\subset V_{i+2},\ fV_i\subset V_{i-2}\quad \forall i.$

(c2) $(ef-fe)|_{V_i} = i\cdot \mathrm{id}_{V_i}\quad \forall i.$

Then, of course, we can define h as $\bigoplus_i i\cdot \mathrm{id}_{V_i}$.

On the level of categories we want to decompose $\mathcal{C}$ into the direct sum of subcategories $\bigoplus_{k\in\mathbb{Z}}\mathcal{C}_k$. As in the example of $\mathcal{C}=\bigoplus_{n\geq 0}\mathbb{F}S_n\text{-mod}$, the decomposition $\mathcal{C}=\bigoplus_{k\in\mathbb{Z}}\mathcal{C}_k$ means that every object M in $\mathcal{C}$ decomposes as $\bigoplus_k M_k$ (with only finitely many nonzero summands), where M_k is an object of $\mathcal{C}_k$, and

$$\mathrm{Hom}_{\mathcal{C}}(M,N)=\bigoplus_k \mathrm{Hom}_{\mathcal{C}_k}(M_k,N_k).$$

1.3) Main example.

We set $\mathcal{C}=\bigoplus_{n\geq 0}\mathbb{F}S_n\text{-mod}$. Fix an element $a\in\mathbb{Z}1\subset\mathbb{F}$ (so it is a residue mod p if $\mathrm{char}\,\mathbb{F}=p$, or an integer if $\mathrm{char}\,\mathbb{F}=0$). Recall, Sections 6.3, 6.4, [RT1], that we can consider a pair of adjoint functors $(\mathrm{Ind}_{n-1}^{n}\bullet)_a:\mathbb{F}S_{n-1}\text{-mod}\leftrightarrows\mathbb{F}S_n\text{-mod}:(\mathrm{Res}_n^{n-1}\bullet)_a$ (the 1st functor is the right adjoint of the 2nd one). We set

$$F_a:=\bigoplus_{n=0}^{\infty}(\mathrm{Ind}_{n-1}^{n}\bullet)_a:\mathcal{C}\leftrightarrows\mathcal{C}:E_a=\bigoplus_{n=0}^{\infty}(\mathrm{Res}_{n-1}^{n}\bullet)_a$$

1.3.1) Weight decomposition of $\mathcal{C}=\bigoplus_n\mathbb{F}S_n\text{-mod}$

Note that the symmetric polynomials in the Jucys-Murphy elements $J_1,\dots,J_n$ are central in $\mathbb{F}S_n$. Indeed, if $\operatorname{char}\mathbb{F}=0$, this is Problem 4 in HW1, but the argument works for any $\mathbb{F}$. Now we can play our usual game: for $M\in\mathbb{F}S_n$-mod we can decompose M into the generalized eigenspaces w.r.t. the central elements. In more detail, choose an unordered n-tuple of elements of $\mathbb{F}$, $\alpha=(a_1,\dots,a_n)$. Let M^α be the subspace of M consisting of all m that are generalized eigenvectors for $P(J_1,\dots J_n)$ w. eigenvalue $P(a_1,\dots,a_n)$, $\forall P\in\mathbb{F}[x_1,\dots,x_n]_{sym}$. Then $M=\bigoplus_\alpha M^\alpha$, where the summation is over all unordered n-tuples of elements of $\mathbb{F}$. Actually, $M^\alpha=\{0\}$ unless $a_i\in\mathbb{Z}1$ $\forall i$: this is because the eigenvalues of $J_1,\dots,J_n$ are in $\mathbb{Z}\cdot 1\subset\mathbb{F}$ (Section 6.4 in [RT1])

Let $\mathbb{F}S_n\text{-mod}^\alpha$ be the full subcategory of $\mathbb{F}S_n$-mod consisting of all M s.t. $M=M^\alpha$. Then $\mathbb{F}S_n\text{-mod}=\bigoplus_\alpha \mathbb{F}S_n\text{-mod}^\alpha$, and so

$$\mathcal{C}=\bigoplus_{n,\alpha}\mathbb{F}S_n\text{-mod}^\alpha.$$

For $b\in\mathbb{Z}\cdot 1$ define $n_b(\alpha)$ to be the number of entries of α equal to b. Then define $wt_a(\alpha):=\delta_{a,0}-2n_a(\alpha)+n_{a-1}(\alpha)+n_{a+1}(\alpha)$. Finally, set $\mathcal{C}_k=\bigoplus_{n,\alpha\,|\,wt_a(\alpha)=k}\mathbb{F}S_n\text{-mod}^\alpha$ so that we indeed have $\mathcal{C}=\bigoplus_k\mathcal{C}_k$.

Exercise: Show that $E_a\mathcal{C}_k\subset\mathcal{C}_{k+2}$, $F_a\mathcal{C}_k\subset\mathcal{C}_{k-2}$, $\forall k$.

1.3.2) Checking $\mathfrak{sl}_2$-relation.

Now we explain why $[E_a],[F_a]$ define a weight representation of $\mathfrak{sl}_2$ in $K_0(\mathcal{C})$ (or its complexification): we check (c2) from Sec 1.2.

Exercise: Check this if char $\mathbb{F}=0$ by recalling that $[E_a]$ sends $[V_\lambda]$ to $[V_\mu]$ if μ is obtained from λ by removing a box of content a or to 0 if no such μ exists, while $[F_a]$ sends $[V_\lambda]$ to $[V_\nu]$ if ν is obtained from λ by adding a box of content a, and 0 if no such ν exists (Cor 5.9 in [RT1] for $[E_a]$, Cor 6.10 in [RT1] for $[F_a]$) We note that μ,ν above are unique if they exist.

To proceed to characteristic p we need some notation motivated by the previous exercise. Let $\mathcal{F}$ (the "Fock space") be the free abelian group with basis labelled by partitions (of all n). We write $|\lambda\rangle$ for the basis vector labelled by a partition λ. For $\tilde{a}\in\mathbb{Z}$, we define operators $e^\infty_{\tilde{a}}, f^\infty_{\tilde{a}}$ on $\mathcal{F}$:

$$e^\infty_{\tilde{a}}|\lambda\rangle=\begin{cases}|\mu\rangle, \text{ if } \mu \text{ is obtained from } |\lambda\rangle \text{ by removing a box of content } \tilde{a},\\ 0, \text{ if no such } \mu \text{ exists.}\end{cases}$$

$f^\infty_{\tilde{a}}|\lambda\rangle$ is defined similarly but we add the box instead of removing it. What the exercise above says is that if we identify $K_0(\bigoplus_{n\geq 0}\mathbb{C}S_n\text{-mod})$ with $\mathcal{F}$ by sending $[V_\lambda]$ to $|\lambda\rangle$, then the operator $[\bigoplus_n \mathrm{Res}^{n-1}_n(\cdot)_{\tilde{a}}]$ is $e^\infty_{\tilde{a}}$, while $[\bigoplus_n \mathrm{Ind}^n_{n-1}(\cdot)_{\tilde{a}}]$ is $f^\infty_{\tilde{a}}$.

Now we proceed to the case of char $\mathbb{F}=p>0$. Pick $a\in\mathbb{F}_p$, which gives the functors E,F. We'll relate $K_0(\mathcal{C})$, $[E]$, and $[F]$ to $\mathcal{F}$, $\sum_{\tilde{a}\equiv a \bmod p} e^\infty_{\tilde{a}}$, and $\sum_{\tilde{a}\equiv a \bmod p} f^\infty_{\tilde{a}}$. The technique we are going to use is the degeneration in K-theory that will allow us to pass from K_0 in char 0 to K_0 in characteristic p.

Let's explain a general setup. Let R be a DVR, $K = \mathrm{Frac}(R)$ & $\Bbbk$ be the residue field. Let $t \in R$ be the parameter so that $K = R[t^{-1}]$, $\Bbbk := R/(t)$. For example, for a prime p, we can take $R = \mathbb{Z}_{(p)}$, the localization at the maximal ideal (p). Then $K = \mathbb{Q}$, $\Bbbk = \mathbb{F}_p$, $t = p$.

Let A_R be an associative R-algebra that is a free finite rank R-module. Then we can form $A_K := K \otimes_R A_R$, $A_\Bbbk := \Bbbk \otimes_R A_R$. Then one has the degeneration map $K_0(A_K\text{-mod}) \xrightarrow{\pi} K_0(A_\Bbbk\text{-mod})$ defined as follows. Take $M^K \in A_K\text{-mod}$. We can choose an A_R-lattice, M^R – an A_R-submodule w. $K \otimes_R M^R \xrightarrow{\sim} M^K$. While M^R is not unique, $[\Bbbk \otimes_R M^R]$ is well-defined, depends only on $[M^K]$ and $[M^K] \longrightarrow [\Bbbk \otimes_R M^R]$ uniquely extends to a group homomorphism, the degeneration map $\pi: K_0(A_K\text{-mod}) \to K_0(A_\Bbbk\text{-mod})$.

The map π may fail to be surjective but for $A_R = RG$, where G is a finite group, it is: Theorem 33, Section 16.1 in Serre's "Linear representations of finite groups."

Take $G = S_n$, $R = \mathbb{Z}_{(p)}$. Note that all irreducible representations of $\mathbb{C}S_n$ are defined over $\mathbb{Q}$ (Section 6.1 of [RT1]). So they are still labelled by the Young diagrams. It allows to identify $K_0(\bigoplus KS_n\text{-mod})$ w. $\mathcal{F}$. We get $\pi: \mathcal{F} \to K_0(\mathcal{C})$ (where we first get to $K_0(\bigoplus \Bbbk S_n\text{-mod})$ and then apply $\mathbb{F} \otimes_\Bbbk \bullet$. The base change $\mathbb{F} \otimes_\Bbbk \bullet$ gives a bijection between the irreducibles of $\Bbbk S_n$ & $\mathbb{F} S_n$: this is by the Wedderburn theorem: a finite skew-field is a field (details of this reduction are left as an exercise).

Consider the operator $e_a^p = \sum_{\tilde{a} \equiv a \bmod p} e_{\tilde{a}}^\infty$ on $\mathcal{F}$. We claim that

$$\pi(e_a^p|\lambda\rangle)=[E_a]\pi(|\lambda\rangle) \qquad (1)$$

Let n be the number of boxes in λ. Consider the KS_{n-1}-submodule

$$V^K_{\lambda,a}:=\bigoplus_{\tilde{a}\equiv a \bmod p} \mathrm{Res}^{n-1}_n(V^K_\lambda)_{\tilde{a}}\subset V^K_\lambda$$

as well as the sum of the remaining eigenspaces, $V^K_{\lambda,\neq a}$. Hence we have the direct sum decomposition $V^K_\lambda=V^K_{\lambda,a}\oplus V^K_{\lambda,\neq a}$ of KS_{n-1}-modules. If V^R_λ is an RS_n-lattice in V^K_λ, then

$$V^R_{\lambda,a}:=V^K_{\lambda,a}\cap V^R_\lambda,\quad V^R_{\lambda,\neq a}:=V^K_{\lambda,\neq a}\cap V^R_\lambda$$

are RS_{n-1}-lattices in $V^K_{\lambda,a}$ & $V^K_{\lambda,\neq a}$. The left hand side of (1) is $[\mathbb{F}\otimes_R V^R_{\lambda,a}]$. We claim that the r.h.s. of (1) is also $[\mathbb{F}\otimes_R V^R_{\lambda,a}]$.

To prove this claim we note that $V^R_{\lambda,a}\oplus V^R_{\lambda,\neq a}$ may fail to be an RS_n-submodule so cannot be used to define the degeneration of $[V^K_\lambda]$. However to define $[E_a]\pi([V^K_\lambda])$ we don't need an RS_n-lattice, it suffices to have a lattice for the subalgebra of RS_n generated by RS_{n-1} & J_n (because for $M\in\mathbb{F}S_n$-mod, $E_a M$ is recovered from the actions of $\mathbb{F}S_{n-1}$ & J_n). And $V^R_{\lambda,a}\oplus V^R_{\lambda,\neq a}$ is such a lattice. Then we just note that

$$E_a(\mathbb{F}\otimes_R(V^R_{\lambda,a}\oplus V^R_{\lambda,\neq a}))=\mathbb{F}\otimes_R V^R_{\lambda,a}$$

and conclude that (1) holds.

One also has the full analog of 1) for the operators f. The starting point here is that, for $M^K\in KS_{n-1}$-mod, an RS_{n-1}-lattice $M^R\subset M^K$ gives rise to the RS_n-lattice $\mathrm{Ind}^n_{n-1}M^R\subset \mathrm{Ind}^n_{n-1}M^K$ and this lattice is stable under the operator J_n^* from Section 6.3 in [RT1]. Details are left as an exercise.

Equation (1) and it's analog for f's reduce the claim that $[E_a], [F_a]$ satisfy the $\mathfrak{sl}_2$-relation i.e.

$$([E_a][F_a]-[F_a][E_a])|_{[\mathcal{C}_k]} = k\, id_{[\mathcal{C}_k]}$$

to the operators e_a^p, f_a^p on $\mathcal{F}$ (where the weight decomposition is as follows: $|\lambda\rangle$ is a weight vector of weight

$$\delta_{a,0} - 2n_a(\lambda) + n_{a-1}(\lambda) + n_{a+1}(\lambda),$$

where, for $b \in \mathbb{F}_p$, $n_b(\lambda)$ is the number of boxes in λ that have content b mod p.

Exercise: 1) Check the relation (c2) from Sec 1.2 for e_a^p, f_a^p.

2) Show that $K_0(\mathcal{C}_k) = \mathbb{Z}\,Span(|\lambda\rangle \mid wt(\lambda) = k)$

3) Deduce the relation for $[E_a], [F_a]$.

1.4) Remark and conclusion.

1) First of all, note that we have operators $[E_a], [F_a]$ on $K_0(\mathcal{C})$ (and e_a^p, f_a^p on $\mathcal{F}$) for each $a \in \mathbb{Z}/p\mathbb{Z}$. Together they give a representation of the Kac-Moody algebra $\hat{\mathfrak{sl}}_p$ - to be mentioned in Lec 17.

2) While the picture explained in this note is, hopefully, fascinating on its own, it is not particularly useful - we haven't really established any tools to study the functors E_a, F_a. This is remedied by throwing in some functor morphisms - to be explained in the 2nd part of this note, to appear after the break.