Representations of $\mathfrak{g}_2$ in categories, part 1.

0) Introduction.

1) Categories & Functors.

0) Much of relatively modern (last 50 years or so) representation theory deals with understanding categories of representations. And a general paradigm to understand all kinds of objects (and forming a foundation of Representation theory, in particular) is to understand their symmetry. In the context of studying representations of semisimple Lie algebras, this idea was very essentially used by Soergel in the 90's, more on this later in the class. Also in the 90's it was envisioned by Igor Frenkel that a categorical symmetry that is a counter part of representations of semisimple Lie algebras (of which $\mathfrak{g}_2$ is the simplest example) should play an important role (in particular, for studying invariants of knots & links - that has been realized since then - and of 3-manifolds - that hasn't). A bit later, it was discovered, e.g. by Chuang and Rouquier that this theory should be useful for Representation theory, including modular representations of symmetric groups. We'll discuss some things about the categorical $\mathfrak{g}_2$-symmetry in this note and subsequent ones.

Rem: This is not a serious remark. In Hegel's philosophy there's the notion of a "triad": thesis $\rightarrow$ antithesis $\rightarrow$ synthesis. In our situation these could be as follows:
Thesis: we want to understand concrete representations (of groups, algebras etc.)

Antithesis: concrete realizations are not important, the structure of a category is.

Synthesis: we'll understand categories by studying their concrete symmetries.

References: Kleshchev's book & Chuang, Rouquier, "Derived equivalences for symmetric groups and \( \mathfrak{sl}_2 \)-categorification"

1) Categories & functors.

We are going to explain the most basic setup. Let \( \mathbb{F} \) be a field.

Let \( \mathcal{C} \) be a category equivalent to a direct sum of categories of finite dimensional representations of finite dimensional associative algebras. A good (and relevant) example is \( \mathcal{C} = \bigoplus_n \mathbb{F} \mathbb{S}_n\)-mod. Every object is a formal finite direct sum of representations of different \( \mathbb{S}_n \)'s. The space of morphisms from \( \bigoplus_i M_i \) to \( \bigoplus_i N_i \) (\( M_i, N_i \in \mathbb{F} \mathbb{S}_i\)-mod) is \( \bigoplus_i \text{Hom}_{\mathbb{S}_i}(M_i, N_i) \).

Rem: More generally, we need \( \mathcal{C} \) to be an \( \mathbb{F} \)-linear abelian category, where every object has finite length, i.e. admits a finite \( \mathcal{J} \)-filtration.

1.1) \( \mathbb{K}_0 \). From a category \( \mathcal{C} \) we can form a free abelian group, its
Grothendieck group, denoted \( K_0(C) \). Namely, it makes sense to speak about simple objects in \( C \), those with proper subobjects. For example, in \( \oplus_{n \geq 0} F_{S_n}\text{-mod} \), the simple objects are exactly the irreducible representations of all symmetric groups.

By definition, \( K_0(C) \) is the free abelian group with basis \( \{L\} \), where \( L \) runs over the set of isomorphism classes of simple objects in \( C \). For an arbitrary object, \( M \in \text{Ob}(C) \), we define \( [M] \in K_0(C) \) as \( [M] = \sum_{i=1}^k [M_1 \oplus M_2 \oplus \cdots \oplus M_k] \), where \( \{0\} = M_0 \subset M_1 \subset \cdots \subset M_k \) is a JH filtration. It is well-defined by the JH theorem.

Now suppose that \( C \) & \( D \) are two categories of the type we consider. Suppose \( F: C \rightarrow D \) is an exact functor. Then we have a group homomorphism \( [F]: K_0(C) \rightarrow K_0(D) \), \( [M] \mapsto [F(M)] \), details are an exercise.

1.2) Categorical \( \mathfrak{S}_2 \)-action, the first attempt.

Now that we can convert categories (of special kind) to abelian groups (and hence, by \( C \otimes \mathbb{Z} \cdot \), to complex vector spaces). We can also convert exact functors to linear maps. So, perhaps, we could try to define an action of \( \mathfrak{S}_2 \) on \( C \) as a triple of exact functors \( E, H, F: C \rightarrow C \) s.t. the operators \( [E], [H], [F] \) on \( K_0(C) \) satisfying the defining relations of \( \mathfrak{S}_2 \). However, this is not what is observed in examples - while the operators \( e, f \) do come from functors, \( h \) doesn't. But if we consider "weight modules," then we don't need \( h \) as an operator.
**Definition:** A weight representation of \( \mathfrak{g}_f \) is a vector space \( V \) together with a direct sum decomposition \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) and two linear operators \( e, f \), s.t.

\[
\begin{align*}
(c1) \quad e V_i & \subset V_{i+1}, \quad f V_i \subset V_{i-1} \quad \forall i, \\
(c2) \quad (e f - f e) V_i &= i \cdot i V_i \quad \forall i.
\end{align*}
\]

Then, of course, we can define \( h \) as \( \bigoplus_{i} i \cdot i V_i \).

On the level of categories, we want to decompose \( C \) into the direct sum of subcategories \( \bigoplus_{k \in \mathbb{Z}} C_k \). As in the example of \( C = \bigoplus_{n \in \mathbb{N}_0} \mathbb{F}S_n\text{-mod} \), the decomposition \( C = \bigoplus_{k \in \mathbb{Z}} C_k \) means that every object \( M \) in \( C \) decomposes as \( \bigoplus_k M_k \) (with only finitely many nonzero summands), where \( M_k \) is an object of \( C_k \), and

\[
\text{Hom}_C(M, N) = \bigoplus_k \text{Hom}_{C_k}(M_k, N_k).
\]

### 1.3) Main example

We set \( C = \bigoplus_{n \in \mathbb{N}_0} \mathbb{F}S_n\text{-mod} \). Fix an element \( a \in \mathbb{Z}^0 \cap F \) (so it is a residue mod \( p \) if \( \text{char} F = p \), or an integer if \( \text{char} F = 0 \)). Recall, Sections 6.3, 6.4, [RTT], that we can consider a pair of adjoint functors \( (\text{Ind}_{n-1}^n)_{a} : \mathbb{F}S_{n-1}\text{-mod} \leftrightarrow \mathbb{F}S_n\text{-mod} : (\text{Res}_{n-1}^n)_{a} \) (the 1st functor is the right adjoint of the 2nd one). We set

\[
F_{a} = \bigoplus_{n=0}^{\infty} (\text{Ind}_{n-1}^n)_{a} : C \leftrightarrow C : E_{a} = \bigoplus_{n=0}^{\infty} (\text{Res}_{n-1}^n)_{a}
\]

### 1.3.1) Weight decomposition of \( C = \bigoplus_{n \in \mathbb{N}_0} \mathbb{F}S_n\text{-mod} \)
Note that the symmetric polynomials in the Jucys-Murphy elements $J_1, \ldots, J_n$ are central in $\mathbb{F} S_n$. Indeed, if $\text{char } \mathbb{F} = 0$, this is Problem 4 in HW1, but the argument works for any $\mathbb{F}$. Now we can play our usual game: for $M \in \mathbb{F} S_n \text{-mod}$ we can decompose $M$ into the generalized eigenspaces w.r.t. the central elements.

In more detail, choose an unordered $n$-tuple of elements of $\mathbb{F}$, $\alpha = (a_1, \ldots, a_n)$. Let $M^\alpha$ be the subspace of $M$ consisting of all $m$ that are generalized eigenvectors for $P(J_1, \ldots, J_n)$ w. eigenvalue $P(a_1, \ldots, a_n)$, $\forall P \in \mathbb{F}[x_1, \ldots, x_n]_{\text{sym}}$. Then $M = \bigoplus \bigoplus M^\alpha$, where the summation is over all unordered $n$-tuples of elements of $\mathbb{F}$. Actually, $M^\alpha = \{0\}$ unless $a_i \in \mathbb{F} \setminus \{0\}$: this is because the eigenvalues of $J_1, \ldots, J_n$ are in $\mathbb{F} \setminus \{0\}$ (Section 6.4 in [RT2]).

Let $\mathbb{F} S_n \text{-mod}^\alpha$ be the full subcategory of $\mathbb{F} S_n \text{-mod}$ consisting of all $M$ s.t. $M = M^\alpha$. Then $\mathbb{F} S_n \text{-mod} = \bigoplus \bigoplus \mathbb{F} S_n \text{-mod}^\alpha$, and so

$$
C = \bigoplus \bigoplus \mathbb{F} S_n \text{-mod}^\alpha.
$$

For $b \in \mathbb{N}$ define $\eta_b(\alpha)$ to be the number of entries of $\alpha$ equal to $b$. Then define $\text{wt}_b(\alpha) := \eta_0 - 2\eta_1(\alpha) + \eta_2(\alpha)$. Finally, set

$$
E^\alpha = \bigoplus \bigoplus \mathbb{F} S_n \text{-mod}^\alpha
$$

so that we indeed have $C = \bigoplus E^\alpha$.

**Exercise:** Show that $E_k^\alpha \subseteq E_{k+1}^\alpha, \quad E_k^\alpha \subseteq E_{k-1}^\alpha, \quad \forall k$.

**13.2) Checking $\delta_k$-relation**

Now we explain why $[E^\alpha], [F^\alpha]$ define a weight representation of $\mathbb{S}_n^\alpha$ in $K_0(C)$ (or its complexification): we check (c2) from Sec 1.2.
Exercise: Check this if char $\mathbb{F} = 0$ by recalling that $[E_{\lambda}]$ sends $[V_{\mu}]$ to $[V_{\nu}]$ if $\mu$ is obtained from $\lambda$ by removing a box of content $a$ or to 0 if no such $\mu$ exists, while $[F_{\lambda}]$ sends $[V_{\mu}]$ to $[V_{\nu}]$ if $\nu$ is obtained from $\lambda$ by adding a box of content $a$, and 0 if no such $\nu$ exists (Cor. 5.9 in [LT7] for $[E_{\lambda}]$, Cor. 6.10 in [LT7] for $[F_{\lambda}]$). We note that $\mu, \nu$ above are unique if they exist.

To proceed to characteristic $p$ we need some notation motivated by the previous exercise. Let $F$ (the "Fock space") be the free abelian group with basis labelled by partitions (of all $n$). We write $|\lambda\rangle$ for the basis vector labelled by a partition $\lambda$. For $\tilde{\alpha} \in \mathbb{N}$, we define operators $e^{\tilde{\alpha}}_{\lambda}, f^{\tilde{\alpha}}_{\lambda}$ on $F$:

$$e^{\tilde{\alpha}}_{\lambda} |\lambda\rangle = \begin{cases} |\mu\rangle, & \text{if } \mu \text{ is obtained from } |\lambda\rangle \text{ by removing a box of content } \tilde{\alpha}, \\ 0, & \text{if no such } \mu \text{ exists.} \end{cases}$$

$$f^{\tilde{\alpha}}_{\lambda} |\lambda\rangle \text{ is defined similarly but we add the box instead of removing it. What the exercise above says is that if we identify } K_{0} (\bigoplus C^{*}_{\mathbb{R}} \text{-mod}) \text{ with } F \text{ by sending } [V_{\lambda}] \text{ to } |\lambda\rangle, \text{ then the operator } [\bigoplus \text{Res}_{n}^{\tilde{\alpha}} (\cdot)] \text{ is } e^{\tilde{\alpha}}_{\lambda}, \text{ while } [\bigoplus \text{Ind}_{n-1}^{\tilde{\alpha}} (\cdot)] \text{ is } f^{\tilde{\alpha}}_{\lambda}.$$

Now we proceed to the case of char $F = p > 0$. Pick $\alpha \in \mathbb{F}_{p}$, which gives the functors $E, F$. We'll relate $K_{0}(C), [E], \text{ and } [F]$ to $F$, $\sum_{\tilde{\alpha} \equiv \alpha \mod p} e^{\tilde{\alpha}}_{\lambda}, \text{ and } \sum_{\tilde{\alpha} \equiv \alpha \mod p} f^{\tilde{\alpha}}_{\lambda}$. The technique we are going to use is the degeneration in $K$-theory that will allow us to pass from $K_{0}$ in char 0 to $K_{0}$ in characteristic $p$.  


Let's explain a general setup. Let $R$ be a DVR, $K = \text{Frac}(R)$ 
& $k$ be the residue field. Let $t \in R$ be the parameter so that $K = \mathbb{R}[t^{-1}]$, $k = R/(t)$. For example, for a prime $p$, we can take $R = \mathbb{Z}_p$, the localization at the maximal ideal $(p)$. Then $K = \mathbb{Q}$, $k = \mathbb{F}_p$, $t = p$.

Let $A_p$ be an associative $R$-algebra that is a free finite 
rank $R$-module. Then we can form $A_k = K \otimes_R A_R$, $A_k = k \otimes_R A_R$. 

Then one has the degeneration map $K_0(A_k \text{-mod}) \to K_0(A_k \text{-mod})$ 
defined as follows. Take $M \in A_k \text{-mod}$. We can choose an $A_k$- 
lattice, $M^k$, an $A_k$-submodule $w. K \otimes_R M^k \to M^k$, while $M^k$ 
is not unique, $[K \otimes_R M^k]$ is well-defined, depends only on $[M^k]$ 
and $[M^k] \to [K \otimes_R M^k]$ uniquely extends to a group homomorphism, 
the degeneration map $\delta_k: K_0(A_k \text{-mod}) \to K_0(A_k \text{-mod})$.

The map $\delta_k$ may fail to be surjective but for $A_p = \mathbb{F}_p$, where 
$\mathbb{F}_p$ is a finite group, it is: Theorem 33, Section 16.1 in Serre's 
"Linear representations of finite groups."

Take $G = S_n$, $R = \mathbb{Z}_p$. Note that all irreducible representations 
of $S_n$ are defined over $\mathbb{Q}$ (Section 6.1 of [RTT]). So they 
are still labelled by the Young diagrams. It allows to identify 
$K_0(\oplus K_{S_n} \text{-mod})$ w. $F$. We get $\delta_k: F \to K_0(C)$ (where we first 
egate get to $K_0(\oplus K_{S_n} \text{-mod})$ and then apply $F \otimes_{k^*}$). The base change 
$F \otimes_{k^*}$ gives a bijection between the irreducibles of $kS_n \& \mathbb{F}S_n$; this is 
by the Wedderburn theorem: a finite skew-field is a field (details of 
this reduction are left as an exercise).

Consider the operator $e_0^0 = \sum_{a \mod \rho} e_0^a$ on $F$. We claim that
\[ \mathcal{M}(c_{\lambda} | \lambda^2) = [E_{\lambda} \mathcal{T}^{-1}(\lambda^2)] \]

Let \( n \) be the number of boxes in \( \lambda \). Consider the \( KS_{n-1} \)-submodule
\[ V_{\lambda, \alpha}^K = \bigoplus_{\alpha \equiv \alpha \mod \rho} Res_{\alpha}^{-1}(V_{\lambda, \alpha}^K) \subset V_{\lambda}^K \]
as well as the sum of the remaining eigenspaces, \( V_{\lambda, \lambda + a}^K \). Hence we have the direct sum decomposition \( V_{\lambda, \alpha}^K = V_{\lambda, \alpha}^K \oplus V_{\lambda, \lambda + a}^K \) of \( KS_{n-1} \)-modules. If \( V_{\lambda}^R \) is an \( RS_n \)-lattice in \( V_{\lambda}^K \), then
\[ V_{\lambda, \alpha}^R = V_{\lambda, \alpha}^K \cap V_{\lambda, \alpha}^R, \quad V_{\lambda, \alpha}^R = V_{\lambda, \alpha}^K \cap V_{\lambda, \alpha}^R \]
are \( RS_{n-1} \)-lattices in \( V_{\lambda, \alpha}^K \) and \( V_{\lambda, \alpha}^K \). The left hand side of (1) is \( [E_{\lambda} \mathcal{T}(V_{\lambda, \alpha}^R)] \). We claim that the r.h.s. of (1) is also \( [E_{\lambda} \mathcal{T}(V_{\lambda, \alpha}^R)] \).

To prove this claim we note that \( V_{\alpha}^R \oplus V_{\lambda, \alpha}^R \) may fail to be an \( RS_n \)-submodule so cannot be used to define the degeneration of \( [V_{\lambda}^K] \). However to define \( [E_{\lambda} \mathcal{T}(V_{\lambda, \alpha}^R)] \) we don’t need an \( RS_n \)-lattice, it suffices to have a lattice for the subalgebra of \( RS_n \) generated by \( RS_{n-1} \) & \( J_n \) (because for \( M \in F_{S_n} \text{-mod}, E_2 M \) is recovered from the actions of \( FS_{n-1} \) & \( J_n \)). And \( V_{\lambda, \alpha}^R \oplus V_{\lambda, \alpha}^R \) is such a lattice. Then we just note that
\[ E_{\lambda}(E \otimes (V_{\alpha}^R \oplus V_{\lambda, \alpha}^R)) = E \otimes V_{\lambda, \alpha}^R \]
and conclude that (1) holds.

One also has the full analog of 1) for the operators \( f \). The starting point here is that, for \( M^K \in KS_{n-1} \text{-mod}, \) an \( RS_{n-1} \)-lattice \( M^R \subset M^K \), gives rise to the \( RS_n \)-lattice \( Ind_n^{n-1} M^R \subset Ind_n^{n-1} M^K \) and this lattice is stable under the operator \( J_n \) from Section 6.3 in [RT1]. Details are left as an exercise.
Equation (1) and its analog for $f$'s reduce the claim that $[E_a], [F_a]$ satisfy the $\mathfrak{sl}_p$-relation i.e.,
\[(E_a)[F_a] - [F_a][E_a])|[e_k]| = k \text{id}|[e_k]|\]
to the operators $e_a^p, f_a^p$ on $F$ (where the weight decomposition is as follows: $|\lambda\rangle$ is a weight vector of weight $\delta_{a_0} - 2n_a(\lambda) + n_{a-1}(\lambda) + n_{a+1}(\lambda)$, where, for $b \in \mathbb{F}_p$, $n_b(\lambda)$ is the number of boxes in $\lambda$ that have content $b \text{ mod } p$.

**Exercise:** 1) Check the relation (c2) from Sec 1.2 for $e_a^p, f_a^p$.
2) Show that $K_b([e_k]) = \mathbb{F} \text{Span}(|\lambda\rangle | \text{wt}(\lambda) = k)$
3) Deduce the relation for $[E_a], [F_a]$.

### 1.4 Remark and conclusion

1) First of all, note that we have operators $[E_a], [F_a]$ on $K_b(C)$ (and $e_a^p, f_a^p$ on $F$) for each $a \in \mathbb{Z}/p\mathbb{Z}$. Together they give a representation of the Kac-Moody algebra $\hat{\mathfrak{sl}}_p$ — to be mentioned in Lec 17.

2) While the picture explained in this note is, hopefully, fascinating on its own, it is not particularly useful — we haven’t really established any tools to study the functors $E_a, F_a$. This is remedied by throwing in some functor morphisms — to be explained in the 2nd part of this note, to appear after the break.