Representations of algebraic groups & their Lie algebras, X.

1) Harish-Chandra (HC) isomorphism for the center of $\mathcal{U}(\mathfrak{g})$.

2) Proof, started.

3) Complements

1.0) Intro.

$\mathbb{F}$: alg. closed char 0 field, $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F})$, $Z:=$ center of $\mathcal{U}(\mathfrak{g})$.

Goal: Describe the algebra Z and understand its action on $\Delta(\lambda)$, and its unique irred. quotient $L(\lambda)$ $(\lambda\in\mathfrak{h}^*)$. Apply this description to prove that every finite dimensional $\mathfrak{g}$-representation is completely reducible.

1.1) Homomorphism $Z\longrightarrow\mathcal{U}(\mathfrak{h})$.

To describe Z we construct an algebra homomorphism $Z\to\mathcal{U}(\mathfrak{h})$. Later we'll see it's injective and describe the image, hence describing Z. This homomorphism will also be used to describe how Z acts on $\Delta(\lambda)$.

Recall: for $\alpha=\varepsilon_i-\varepsilon_j$ ($i<j$, a positive root) we write $f_\alpha:=E_{ji}$, $e_\alpha=E_{ij}$. For $i=1,\dots,n-1$, $h_i:=E_{ii}-E_{i+1,i+1}$; $N=\frac{n(n-1)}{2}$, $\beta_1,\dots,\beta_N$ - all positive roots.

PBW Thm: $\mathcal{U}\mathfrak{g}$ has basis
$$\prod_{j=1}^{N} f_{\beta_j}^{k_j}\prod_{i=1}^{n-1} h_i^{\ell_i}\prod_{j=1}^{N} e_{\beta_j}^{m_j}\qquad(1)$$

$\mathfrak{g}$, hence $\mathfrak{h}$, acts on $\mathcal{U}(\mathfrak{g})$ by ad: $\operatorname{ad}(x)a:=[x,a]$ $(x\in\mathfrak{h},\ a\in\mathcal{U}\mathfrak{g})$.

Exercise: (1) is a weight vector of weight $\sum_{j=1}^{N}(m_j-k_j)\beta_j$ (hint: $\forall x\in\mathfrak{h}$, $a,b\in\mathcal{U}(\mathfrak{g})$, have $[x,ab]=[x,a]b+a[x,b]$).

Now we define a map $z \mapsto HC_z : Z \longrightarrow U(\mathfrak{h})$. By definition, HC_z is the sum of all monomials in the expansion of z in (1) that only have h_i's.

Example: for $C = \frac{1}{2}h^2 + h + 2fe \in Z \subset U(\mathfrak{sl}_2) \Rightarrow HC_C = \frac{1}{2}h^2 + h$.

Note that all monomials in the expansion of $z - HC_z$ must have $k_j > 0$, $m_{j'} > 0$ for some j, j': $[x, z] = 0$, $\forall x \in \mathfrak{h}$, $\Rightarrow$ z has weight 0, therefore every monomial in z must have weight 0. So, $HC_z \in U(\mathfrak{h})$ satisfies

$$z = HC_z + \sum_{j=1}^{N} ?\, e_{\beta_j}. \qquad (2)$$

Note that $\mathfrak{h}$ is an abelian Lie algebra $\Rightarrow U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{F}[\mathfrak{h}^*]$. So we can view HC_z as a polynomial on $\mathfrak{h}^*$.

Proposition: 1) $\forall\, z \in Z$, $\lambda \in \mathfrak{h}^*$, z acts on $\Delta(\lambda)$ & $L(\lambda)$ by $HC_z(\lambda)$.

2) $z \mapsto HC_z$ is an algebra homomorphism.

Proof: 1) Have $\Delta(\lambda) = U(\mathfrak{g})v_\lambda$ & z commutes w. $U(\mathfrak{g})$. So it's enough to show $zv_\lambda = HC_z(\lambda)v_\lambda$. But $e_\alpha v_\lambda = 0$ $\forall$ positive roots α, so (2) $\Rightarrow$ $zv_\lambda = HC_z(\lambda)v_\lambda$. The claim for $L(\lambda)$ follows b/c $\Delta(\lambda) \twoheadrightarrow L(\lambda)$.

2) $z \mapsto HC_z$ is $\mathbb{F}$-linear by construction. By 1), $HC_{z_1 z_2}(\lambda) = HC_{z_1}(\lambda) HC_{z_2}(\lambda)$ $\forall\, \lambda \in \mathfrak{h}^*$, $z_1, z_2 \in Z$. So $HC_{z_1 z_2} = HC_{z_1} HC_{z_2}$. $\square$

scalar by which z_1, z_2 acts on $\Delta(\lambda)$

1.2) Harish-Chandra isomorphism.

Proposition in Sec 1.1 & Sec 1.2 of Lec 13 have an important consequence.

For $i=1,\dots,n-1$, define $s_i\cdot\lambda = \lambda - (\langle\lambda,h_i\rangle+1)\alpha_i$ so that $s_i\cdot$ is an affine map $\mathfrak{h}^*\to\mathfrak{h}^*$ ($s_i\cdot\lambda=\lambda_i'$ in the notation of Lec 13).

Proposition $\forall z\in Z$, $\lambda\in\mathfrak{h}^*$ have $HC_z(\lambda)=HC_z(s_i\cdot\lambda)$.

Proof: Case 1: $\langle\lambda,h_i\rangle\in\mathbb{Z}_{\geq 0}$. By Sec 1.2 of Lec 13, $\exists$ nonzero $\mathcal{U}(\mathfrak{g})$-linear homomorphism $\Delta(s_i\cdot\lambda)\to\Delta(\lambda)\Rightarrow$ scalars of actions of $z\in\mathcal{U}(\mathfrak{g})$ on $\Delta(s_i\cdot\lambda)$, $\Delta(\lambda)$ coincide. By Prop 1, $HC_z(\lambda)=HC_z(s_i\cdot\lambda)$.
Case 2: general. The locus $\{\lambda\in\mathfrak{h}^* \mid \langle\lambda,h_i\rangle\in\mathbb{Z}_{\geq 0}\}$ is a countable union of hyperplanes: $\{\lambda\in\mathfrak{h}^*\mid\langle\lambda,h_i\rangle=m\}$ for $m\in\mathbb{Z}_{\geq 0}$. Any polynomial vanishing on such locus is identically 0. Apply this to the polynomial $\lambda\mapsto HC_z(\lambda)-HC_z(s_i\cdot\lambda)$ & finish the proof. $\square$

Example: For $\mathfrak{sl}_2$: $\mathfrak{h}\simeq\mathbb{C}$ w. $h\leftrightarrow 1 \rightsquigarrow \mathfrak{h}^*\simeq\mathbb{C}$ w. $\alpha=2$, $\rho=1$, $s\cdot\lambda=-\lambda-2$. Since $HC_C=\frac{1}{2}h^2+h$, we get $HC_C(\lambda)=\frac{1}{2}\lambda^2+\lambda=HC_C(-\lambda-2)$.

In fact, $\lambda\mapsto s_i\cdot\lambda$ extends to an action of the Weyl group $W(=S_n)$ on $\mathfrak{h}^*$. Set $\rho=\frac{1}{2}\sum_{i<j}(\varepsilon_i-\varepsilon_j)=\sum_{i=1}^{n}\left(\frac{n+1}{2}-i\right)\varepsilon_i\in\mathfrak{h}^*$ so that $\langle\rho,h_i\rangle=1$
$\Rightarrow s_i\rho=\rho-\alpha$. Then $s_i(\lambda+\rho)-\rho=\lambda+\rho-\langle\lambda+\rho,\alpha_i^\vee\rangle-\rho=\lambda-(\langle\lambda,h_i\rangle+1)\alpha_i=s_i\cdot\lambda$.

Definition: The *shifted action* of W on $\mathfrak{h}^*$ is given by $w\cdot\lambda:=w(\lambda+\rho)-\rho$.

Consider the subalgebra $\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}=\{f\in\mathbb{F}[\mathfrak{h}^*]\mid f(w\cdot\lambda)=f(\lambda),\ \forall\lambda\in\mathfrak{h}^*, w\in W\}$ of invariant polynomials. Since the elements s_i generate W, Proposition above implies $HC_z \in \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$ $\forall z \in Z$. The following will be proved next time.

Thm (Harish-Chandra) $z \mapsto HC_z : Z \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$.

Corollary: For $\lambda, \mu \in \mathfrak{h}^*$ TFAE

(1) $\lambda \in W \cdot \mu$.

(2) $HC_z(\lambda) = HC_z(\mu)$, $\forall z \in Z$.

Proof: (1) $\Rightarrow$ (2) is a direct consequence of the theorem. (2) $\Rightarrow$ (1) becomes: if $f(\lambda) = f(\mu)$ $\forall f \in \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$, then $\lambda \in W \cdot \mu$. This is exercise (hint: find a polynomial f that is 1 on $W \cdot \lambda$, 0 on $W \cdot \mu$ and average w.r.t. W-action: $f \mapsto \frac{1}{|W|} \sum_{w \in W} f(w \cdot ?)$).

1.3) Application: complete reducibility.

Thm: Every finite dimensional representation of $\mathfrak{g}$ is completely reducible.

Proof: Let $\lambda, \mu \in \Lambda^+$. Then $\lambda + \rho, \mu + \rho$ are strictly decreasing so $\lambda \in W \cdot \mu$ ($\Leftrightarrow$ $\lambda + \rho \in W(\mu + \rho)$ for the usual action $\Leftrightarrow$ $\lambda + \rho$ is obtained from $\mu + \rho$ by permutation) implies $\lambda = \mu$. So, thx to Corollary in Sec 1.3, if $\lambda \neq \mu$ $\exists z \in Z$ acting on $L(\lambda), L(\mu)$ by different scalars.

Once we know we can prove the complete reducibility of finite dimensional $\mathfrak{g}$-representations similarly to the $\mathfrak{sl}_2$-case. There are no new ideas just technicalities, the proof is in the complement section. $\square$

The following establishes some claims made in Lec 13.

Corollary: 1) Every nonzero finite dimensional quotient of a Verma module is irreducible. In particular, $\tilde{L}(\lambda) \xrightarrow{\sim} L(\lambda)$ (see Sec 1.3 of Lec 13).

2) Let $\lambda \in \Lambda^+$, U a finite dimensional $\mathfrak{g}$-representation, $u \in U_\lambda$ s.t. $\mathfrak{n}u = 0$ ($\Leftrightarrow e_\alpha u = 0$, $\forall$ positive root α). Then $U(\mathfrak{g})u \subset U$ is irreducible.

Proof: Any quotient, M, of $\Delta(\lambda)$ has the unique irreducible quotient, $L(\lambda)$. So, M is completely reducible $\Leftrightarrow$ M is irreducible. Applying Theorem, get (1).

To prove (2) note that $\Delta(\lambda) \twoheadrightarrow U(\mathfrak{g})u$, compare to proof of Proposition in Sec 1.1 of Lec 13. So, (1) $\Rightarrow$ (2). $\square$

Rem: We don't need the full power of HC isomorphism to prove the complete reducibility - there are more elementary proofs, e.g. Sec 6.3 in [K] or Sec 6.5 in [B]. We will essentially use the theorem when we compute the character of $L(\lambda)$, $\lambda \in \Lambda_+$.

1.4) Algebra $\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$

Consider the affine isomorphism $\tau: \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}^*$, $\lambda \mapsto \lambda + \rho$ so that $\tau(w\cdot\lambda) = w\,\tau(\lambda)$. So τ gives rise to an isomorphism $\tau: \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)} \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^W$. Let's describe the target. Embed $\mathfrak{h}^* \hookrightarrow \mathbb{F}^n$ as refl_n. Define $p_k \in \mathbb{F}[\mathfrak{h}^*]^W$ by $p_k(x_1, \dots, x_n) = \sum_{i=1}^n x_i^k$ for $k > 1$ ($p_1 = 0$).

Lemma: $\mathbb{F}[\mathfrak{h}^*]^W$ is the algebra of polynomials in $p_2, \dots, p_n$.

Proof: exercise - note that we are essentially dealing w. the algebra of symmetric polynomial.

Exercise: $Z\subset U(\mathfrak{sl}_2)$ is generated by C.

2) Proof, started.

2.1) Z vs $U(\mathfrak{g})^G$

To establish the HC isomorphism, we'll need an alternative description of Z. Let G be a connected algebraic group w. Lie algebra $\mathfrak{g}$. Recall, Sec 1.2 of Lec 10, that G acts on $U(\mathfrak{g})$ by algebra automorphisms $\rightsquigarrow$ the subalgebra $U(\mathfrak{g})^G\subset U(\mathfrak{g})$ of invariants.

Lemma: $Z=U(\mathfrak{g})^G$.

Proof: $Z=\{a\in U(\mathfrak{g}) \mid ad(x)a=0\ \forall x\in\mathfrak{g}\}$. We write $\mathbb{F}$ for the trivial representation of $\mathfrak{g}$ or of G. Then

$$\begin{array}{ccc}
Z & \ni & \varphi(1)\\
\sim\downarrow & & \updownarrow\\
\operatorname{Hom}_{\mathfrak{g}}(\mathbb{F},U(\mathfrak{g})) & \ni & \varphi\\
\| & & \\
\operatorname{Hom}_G(\mathbb{F},U(\mathfrak{g})) & \ni & \varphi\\
\sim\downarrow & & \updownarrow\\
U(\mathfrak{g})^G & \ni & \varphi(1)
\end{array}$$

By Thm 2 in Sec 1.3 of Lec 7.

$\square$

3) Complements.

Here are some details for proving Theorem in Sec 1.4.

- Decomposition into "infinitesimal blocks": Let V be a $\mathfrak{g}$-representation (not necessarily finite dimensional). Let $\chi: Z\to\mathbb{F}$ be an algebra homomorphism. Set

$V^\chi = \{v \in V \mid \forall z \in Z\ \exists m > 0 \text{ s.t. } (z - \chi(z))^m v = 0\}$

This is a $U(\mathfrak{g})$-submodule in V. If V is finite dimensional, then $V = \bigoplus_\chi V^\chi$. Moreover, z acts by $\chi(z)$ on every irreducible constituent of V^χ. It follows that $\chi(z) = HC_z(\lambda)$ for some $\lambda \in \Lambda^+$ whenever $V^\chi \neq \{0\}$. Moreover, by the observation in the proof of the theorem in Sec 1.4, in this case $L(\lambda)$ is the unique irreducible constituent of V^χ.

So assume $V = V^\chi \Leftrightarrow V$ is filtered by $L(\lambda)$ w. $\chi(z) = HC_z(\lambda)$. Then $L(\lambda) \otimes V_\lambda \xrightarrow{\sim} V$, the proof repeats that in Sec 1.3 of Lec 9. Details are left as an exercise.