Representations of algebraic groups & their Lie algebras, X.

1) Harish-Chandra (HC) isomorphism for the center of $U(g)$.
2) Proof, started.
3) Complements

1.0) Intro.

$F$: alg. closed char 0 field, $g = L_k^r(F)$, $Z$: = center of $U(g)$.

Goal: Describe the algebra $Z$ and understand its action on $\Delta(\lambda)$, and its unique irreducible quotient $\mathbb{L}(\lambda)$ ($\lambda \in \mathfrak{k}^*$). Apply this description to prove that every finite dimensional $g$-representation is completely reducible.

1.1) Homomorphism $Z \rightarrow U(g)$

To describe $Z$ we construct an algebra homomorphism $Z \rightarrow U(g)$. Later we'll see it's injective and describe the image, hence describing $Z$. This homomorphism will also be used to describe how $Z$ acts on $\Delta(\lambda)$.

Recall: for $d = e_i - e_j$ ($i \neq j$, a positive root) we write $f_d = E_{ij}, e_d = E_{ji}$. For $i = 1, \ldots, n-1$, $h_i = E_{ii} - E_{i,i+1}$; $N = \frac{1}{2} \beta_1, \beta_2, \ldots, \beta_N$ - all positive roots.

PBW Thm: $U(g)$ has basis:

\[
\prod_{j=1}^N f_{p_j}^k h_i^l e_i^m \quad (i) \tag{1}
\]

$g$, hence $k$, acts on $U(g)$ by ad: $\text{ad}(a)x = [x,a]$ ($x \in g$, $a \in U(g)$).

Exercise: (1) is a weight vector of weight $\sum_{j=1}^N (m_j - k_j) \beta_j$ (hint: if $x \in \mathfrak{k}$, $a, b \in U(g)$, have $[x, ab] = [x,a]b + a[x,b])$. 

1
Now we define a map $z \mapsto HZ_{\ast} : Z \rightarrow U(\mathfrak{g})$. By definition, $HZ_{\ast}$ is the sum of all monomials in the expansion of $z$ in (1) that only have $h_i$'s.

**Example**: for $C = \frac{1}{2} h_i^2 + h_j + 2t e \in Z \subset U(\mathfrak{g})_2 \Rightarrow HZ_{C} = \frac{1}{2} h_i^2 + h_j$.

Note that all monomials in the expansion of $z - HZ_{\ast}$ must have $k_i > 0, m_j > 0$ for some $j, j' : [x, t] = 0, \forall x \in \mathfrak{h}_i$, $\Rightarrow z$ has weight 0, therefore every monomial in $z$ must have weight 0. So $HZ_{\ast} \in U(\mathfrak{g})$ satisfies

$$z = HZ_{\ast} + \sum_{j=1}^{N} \varepsilon_j e_j. \quad (2)$$

Note that $\mathfrak{g}$ is an abelian Lie algebra $\Rightarrow U(\mathfrak{g}) = S(\mathfrak{g}) = \mathfrak{g}[t^\pm]$. So we can view $HZ_{\ast}$ as a polynomial on $\mathfrak{g}^\ast$.

**Proposition**: 1) $\forall z \in Z, \lambda \in \mathfrak{g}^\ast, z$ acts on $\Delta(\lambda)$ & $\mathcal{L}(\lambda)$ by $HZ_{\ast}(\lambda)$.

2) $z \mapsto HZ_{\ast}$ is an algebra homomorphism.

**Proof**: 1) Have $\Delta(\lambda) = U(\mathfrak{g}) \cdot v_{\lambda}$ & $z$ commutes w. $U(\mathfrak{g})$. So it's enough to show $z v_{\lambda} = HZ_{\ast}(\lambda) v_{\lambda}$. But $z v_{\lambda} = 0 \lor$ positive roots $\lambda$, so (2) $\Rightarrow z v_{\lambda} = HZ_{\ast}(\lambda) v_{\lambda}$.

The claim for $\mathcal{L}(\lambda)$ follows $y / \Delta(\lambda) \rightarrow \mathcal{L}(\lambda)$.

2) $z \mapsto HZ_{\ast}$ is $\mathbb{C}[\mathfrak{g}]$-linear by construction. By 1), $HZ_{z_1 z_2}(\lambda) = HZ_{z_1}(\lambda) HZ_{z_2}(\lambda)$

$\forall \lambda \in \mathfrak{g}^\ast, z_1, z_2 \in Z$. So $HZ_{z_1 z_2} = HZ_{z_1} HZ_{z_2}$. Scalar by which $z_1, z_2$ acts on $\Delta(\lambda)$.$\square$

1.2) Harish-Chandra isomorphism.

Proposition in Sec 1.1 & Sec 1.2 of Lec 13 have an important consequence.
For $i=1,\ldots,n-1$, define $s_i \cdot \lambda = \lambda - (\langle \lambda, h_i \rangle + 1) d_i$ so that $s_i$ is an affine map $\mathfrak{g}^* \to \mathfrak{g}^*$. So that $s_i$ is in the notation of Sec. 13.

**Proposition** \( \forall z \in \mathbb{Z}, \lambda \in \mathfrak{g}^* \) have \( HC_z(\lambda) = HC_z(s_i \cdot \lambda) \).

**Proof:** Case 1: \( \langle \lambda, h_i \rangle \in \mathbb{Z}_{\geq 0} \). By Sec. 1.2 of Sec. 13, \( \exists \) nonzero \( U(\mathfrak{g}) \)-linear homomorphism \( \Delta(s_i \cdot \lambda) \to \Delta(\lambda) \) \( \Rightarrow \) scalars of actions of \( z \in U(\mathfrak{g}) \) on \( \Delta(s_i \cdot \lambda), \Delta(\lambda) \) coincide. By Prop 1, \( HC_z(\lambda) = HC_z(s_i \cdot \lambda) \).

Case 2: general. The locus \( \{ \lambda \in \mathfrak{g}^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z}_{\geq 0} \} \) is a countable union of hyperplanes: \( \{ \lambda \in \mathfrak{g}^* \mid \langle \lambda, h_i \rangle = m \} \) for \( m \in \mathbb{Z}_{\geq 0} \). Any polynomial vanishing on such locus is identically 0. Apply this to the polynomial \( \lambda \mapsto HC_z(\lambda) - HC_z(s_i \cdot \lambda) \) & finish the proof. \( \square \)

**Example:** For \( \mathfrak{g} \): \( \mathfrak{g} \cong \mathbb{C} \) w. \( h \leftrightarrow 1 \to \mathfrak{g}^* \cong \mathbb{C} \) w. \( x = 2, p = 1, s_i \cdot \lambda = -\lambda - 2 \). Since \( HC_z = \frac{1}{2} h^2 + h \), we get \( HC_z(\lambda) = \frac{1}{2} \lambda^2 + \lambda = HC_z(-\lambda - 2) \).

In fact, \( \lambda \mapsto s_i \cdot \lambda \) extends to an action of the Weyl group \( W = S_n \) on \( \mathfrak{g}^* \). Set \( p = \frac{1}{2} \sum_{i=1}^n (x_i \cdot x_j) = \sum_{i=1}^n \left( \frac{\rho - i}{2} \right) x_i \in \mathfrak{g}^* \) so that \( \langle p, h_i \rangle = 1 \) \( \Rightarrow s_i \cdot p = p - d_i \). Then \( s_i(\lambda + p) = \lambda + p - \langle \lambda + p, x_i \rangle = \lambda - (\langle \lambda, h_i \rangle + 1) d_i = s_i(\lambda) \).

**Definition:** The shifted action of \( W \) on \( \mathfrak{g}^* \) is given by \( w \cdot \lambda = w(\lambda + p) \).

Consider the subalgebra \( \mathcal{V}[\mathfrak{g}^*(W)] = \{ f \in \mathcal{V}[\mathfrak{g}^*] \mid f(w \cdot \lambda) = f(\lambda), \forall \lambda \in \mathfrak{g}^*, \forall w \in W \} \) of invariant polynomials. Since the elements \( s_i \) generate \( W \), Proposition above
implies \( HC_2 \in \mathbb{F}[\mathbb{F}^*]^{(W_z)} \forall z \in \mathbb{Z} \). The following will be proved next time.

Thm (Harish-Chandra) \( z \mapsto HC_2 : \mathbb{Z} \to \mathbb{F}[\mathbb{F}^*]^{(W_z)} \)

**Corollary:** For \( \lambda, \mu \in \mathbb{F}^* \) TFAE

1. \( \lambda \in W_1 \mu \).
2. \( HC_2 (\lambda) = HC_2 (\mu) \forall z \in \mathbb{Z}. \)

Proof: (1) \( \Rightarrow \) (2) is a direct consequence of the theorem. (2) \( \Rightarrow \) (1) becomes:

if \( f(\lambda) = f(\mu) \forall f \in \mathbb{F}[\mathbb{F}^*]^{(W_z)} \), then \( \lambda \in W_1 \mu \). This is exercise (hint: find a polynomial \( f \) that is 1 on \( W_1 \lambda \), 0 on \( W_1 \mu \) and average w.r.t. \( W \)-action:

\[ f \mapsto \frac{1}{|W|} \sum_{w \in W} f(w^{-1} \cdot \cdot ). \]

1.3) **Application: complete reducibility.**

**Thm:** Every finite dimensional representation of \( g \) is completely reducible.

Proof: Let \( \lambda, \mu \in \Lambda^+. \) Then \( \lambda + p, \mu + p \) are strictly decreasing so \( \lambda \in W_1 \mu \) \( \Leftrightarrow \lambda + p \in W_1 (\mu + p) \) for the usual action \( \Leftrightarrow \lambda + p \) is obtained from \( \mu + p \) by permutation) implies \( \lambda = \mu \). So, thx to Corollary in Sec 1.3, if \( \lambda \neq \mu \) \( \exists z \in \mathbb{Z} \) acting on \( L(\lambda), L(\mu) \) by different scalars.

Once we know we can prove the complete reducibility of finite dimensional \( g \)-representations similarly to the \( \mathfrak{sl}_2 \)-case. There are no new ideas, just technicalities, the proof is in the complement section. \( \square \)

The following establishes some claims made in Sec 13.
Corollary: 1) Every nonzero finite dimensional quotient of a Verma module is irreducible. In particular, \( \overline{L}(\lambda) \approx L(\lambda) \) (see Sec 1.3 of Lec 13).

2) Let \( \lambda \in \Lambda^+ \), \( U \) a finite dimensional \( g \)-representation, \( u \in U \) s.t. 
\[ \mu \neq 0 \quad (\Leftrightarrow \epsilon \mu = 0, \neq \text{positive root } \alpha) \]. Then \( U(g)u \subseteq U \) is irreducible.

Proof: Any quotient, \( M \), of \( \Delta(\lambda) \) has the unique irreducible quotient, \( L(\lambda) \). So, \( M \) is completely reducible \( \Leftrightarrow M \) is irreducible. Applying Theorem, get (1).

To prove (2), note that \( \Delta(\lambda) \rightarrow U(g)u \), compare to proof of Proposition in Sec 1.1 of Lec 13. So, (1) \( \Rightarrow \) (2).

Remark: We don't need the full power of HC isomorphism to prove the complete reducibility - there are more elementary proofs, e.g. Sec 6.9 in [K] or Sec 6.5 in [B]. We will essentially use the theorem when we compute the character of \( L(\lambda), \lambda \in \Lambda^+ \).

1.4) Algebra \( \mathbb{F}[\mathfrak{g}^*]^W \)

Consider the affine isomorphism \( \tau: \mathfrak{g}^* \approx \mathfrak{g}^*, \lambda \mapsto \lambda - \rho \) so that \( \tau(W \cdot \lambda) = W \tau(\lambda) \). So \( \tau \) gives rise to an isomorphism \( \tau^*: \mathbb{F}[\mathfrak{g}^*]^W \rightarrow \mathbb{F}[\mathfrak{g}^*]^W \).

Let's describe the target. Embed \( \mathfrak{g}^* \rightarrow \mathbb{F}^n \) as \( \mathfrak{g}^* \cdot \mathfrak{g} \). Define \( p_k \in \mathbb{F}[\mathfrak{g}^*]^W 
\] by 
\[ p_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k \quad \text{for } k \geq 1 \text{ } (p_0 = 0). \]

Lemma: \( \mathbb{F}[\mathfrak{g}^*]^W \) is the algebra of polynomials in \( p_1, \ldots, p_n \).

Proof: Exercise - note that we are essentially dealing w the algebra of symmetric polynomials.
Exercise: $Z = U(\mathfrak{g}_2^\mathbb{C})$ is generated by $C$.

2) Proof started.

2.1) $\mathfrak{z}$ vs $U(\mathfrak{g})^G$

To establish the HC isomorphism, we’ll need an alternative description of $Z$. Let $G$ be a connected algebraic group w. Lie algebra $\mathfrak{g}$. Recall, Sec 1.2 of Lec 10, that $G$ acts on $U(\mathfrak{g})$ by algebra automorphisms $\to$ the subalgebra $U(\mathfrak{g})^G \subset U(\mathfrak{g})$ of invariants.

Lemma: $Z = U(\mathfrak{g})^G$.

Proof: $Z = \{ a \in U(\mathfrak{g}) \mid \text{ad}(a) a = 0 \forall x \in \mathfrak{g} \}$. We write $\mathbb{1}$ for the trivial representation of $\mathfrak{g}$ or of $G$. Then

$$Z \cong \mathbb{1}$$

By Thm 2 in Sec 1.3 of Lec 7.

3) Complements.

Here are some details for proving Theorem in Sec 1.4.

- Decomposition into "infinitesimal blocks": Let $V$ be a $\mathfrak{g}$-representation (not necessarily finite dimensional). Let $X: Z \to \mathbb{1}$ be an algebra homomorphism. Set
This is a \( U(q) \)-submodule in \( V \). If \( V \) is finite dimensional, then \( V = \bigoplus V^x \). Moreover, \( z \) acts by \( \chi(z) \) on every irreducible constituent of \( V^x \). It follows that \( \chi(z) = H\zeta(x) \) for some \( \lambda \in \Lambda^+ \) whenever \( V^x \neq \{0\} \). Moreover, by the observation in the proof of the theorem in Sec 1.4, in this case \( L(\lambda) \) is the unique irreducible constituent of \( V^x \).

So assume \( V = V^x \iff V \) is filtered by \( L(\lambda) \) w. \( \chi(z) = H\zeta(x) \). Then \( L(\lambda) \otimes V_x \hookrightarrow V \), the proof repeats that in Sec 1.3 of Lec 9. Details are left as an exercise.