Representations of algebraic groups & their Lie algebras, XI.

1) Chevalley restriction theorem.
2) Proof of HC isomorphism.
3) Complements.

1) In the last lecture we have stated the Harish-Chandra isomorphism theorem for the center $Z = U(\mathfrak{g})^G$ of $U(\mathfrak{g})$ ($G = SL_n$), to be recalled below.

The first step is to prove an analog of HC isomorphism in an easier setting. Consider the symmetric algebra $S(\mathfrak{g})$; G acts by automorphisms $\rightsquigarrow$ subalgebra $S(\mathfrak{g})^G$ of invariants. We'll establish an isomorphism $S(\mathfrak{g})^G \to S(\mathfrak{h})^W$, where $W (\simeq S_n)$ is the Weyl group. This is known as the Chevalley restriction theorem.

We start with an exercise (compare to the case of $\mathfrak{sl}_2$ in Sec 1.3 of Lec 10). Consider the adjoint action of G on $\mathfrak{g}$: $g.x = gxg^{-1}$.

Exercise: Show that $(x,y) := tr(xy)$ defines a G-invariant non-degenerate symmetric bilinear form.

This gives rise to a G-equivariant isomorphism $\mathfrak{g} \to \mathfrak{g}^*$, $x \mapsto tr(x\cdot)$, hence an algebra isomorphism $S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{g}^*)^G = \mathbb{F}[\mathfrak{g}]^G$, where, explicitly

$$\mathbb{F}[\mathfrak{g}]^G = \{F \in \mathbb{F}[\mathfrak{g}] \mid F(x) = F(gxg^{-1}), \forall x \in \mathfrak{g}, g \in G\}$$

We have the restriction homomorphism res: $\mathbb{F}[\mathfrak{g}] \to \mathbb{F}[\mathfrak{h}]$, $F \mapsto F|_{\mathfrak{h}}$.

Example: Pick $k \geq 1$. Set $F_k(x) = tr(x^k) \in \mathbb{F}[\mathfrak{g}]$. Then $res(f) = p_k$, k-th power symmetric polynomial, $p_k(diag(x_1,\ldots x_n)) = \sum_{i=1}^n x_i^k$.

The action of $S_n = W$ on $\mathfrak{h}$, its reflection representation, gives rise to $W \curvearrowright \mathbb{F}[\mathfrak{h}]$ by algebra automorphisms. The algebra of invariants, $\mathbb{F}[\mathfrak{h}]^W$, is $\mathbb{F}[p_2, \dots, p_n]$, compare to Sec 1.4 of Lec 14.

Thm ($\mathfrak{sl}_n$-special case of the Chevalley restriction thm)
$res: \mathbb{F}[\mathfrak{g}] \longrightarrow \mathbb{F}[\mathfrak{h}]$ restricts to $\mathbb{F}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}]^W$.

Proof: We need to show:

(a) $res(\mathbb{F}[\mathfrak{g}]^G) \subset \mathbb{F}[\mathfrak{h}]^W$

(b) $\mathbb{F}[\mathfrak{h}]^W \subset res(\mathbb{F}[\mathfrak{g}]^G)$.

(c) $res|_{\mathbb{F}[\mathfrak{g}]^G}$ is injective.

(a): Exercise: Pick $w \in W$ and a "monomial matrix" $M_w \in SL_n(\mathbb{F})$, $M_w = (m_{ij})$ w. $m_{ij} \neq 0$ only if $i = w(j)$. Then

$$M_w \operatorname{diag}(x_1, \dots, x_n) M_w^{-1} = \operatorname{diag}(x_{w(1)}, \dots x_{w(n)}).$$

Now take $F \in \mathbb{F}[\mathfrak{g}]^G$. Of course, $F(\operatorname{diag}(x_1, \dots x_n)) = F(M_w \operatorname{diag}(x_1, \dots x_n) M_w^{-1}) =$
$= F(\operatorname{diag}(x_{w(1)}, \dots, x_{w(n)})) \Rightarrow res(F) \in \mathbb{F}[\mathfrak{h}]^W$.

(b): $\mathbb{F}[\mathfrak{h}]^W$ is generated by $p_2, \dots, p_n$. By the example above, $p_i = res(F_i)$ w. $F_i = tr(x^i) \in \mathbb{F}[\mathfrak{g}]^G$. (b) follows.

(c): Let $F \in ker(res) \cap \mathbb{F}[\mathfrak{g}]^G$. It vanishes on every diagonal - and b/c of G-invariance, on every <u>diagonalizable</u> matrix. Those form a Zariski dense subset in $\mathfrak{g}$. So $F = 0$. $\square$

Rem: We can identify $\mathfrak{h}$ w $\mathfrak{h}^*$ using the trace form as well. So we get the map $res: S(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{g}^*) \xrightarrow{res} S(\mathfrak{h}^*) \xrightarrow{\sim} S(\mathfrak{h})$, an algebra homomorphism extended from $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^* \xrightarrow{res} \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$: $x \in \mathfrak{g} \mapsto res(x)$ w. $tr(xy) = tr(res(x)y)$ $\forall y \in \mathfrak{h}$. It follows that on the basis e_α, f_α, h_i, we have $res(h_i) = h_i$ & $res(e_\alpha) = res(f_\alpha) = 0$ (e.g. $tr(e_\alpha y) = 0$ $\forall y \in \mathfrak{h}$).

2) Proof of HC isomorphism

We'll deduce the Harish-Chandra isomorphism from the Chevalley thm.

2.1) $sym: S(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$

We want to compare $Z = U(\mathfrak{g})^G$ to $S(\mathfrak{g})^G$. We'll do so by constructing a G-linear vector space isomorphism $S(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$.

Consider a map $\mathfrak{g}^k \longrightarrow U(\mathfrak{g})$, $(\xi_1, \dots \xi_k) \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)}$. This map is multilinear and symmetric (the image stays the same if we permute the arguments) so gives a unique linear map $sym: S^k(\mathfrak{g}) \to U(\mathfrak{g})$, w. $\xi_1 \cdots \xi_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)}$. Extend it to $S(\mathfrak{g})$ by linearity.

Lemma: The map sym has the following properties:

1) $sym(\xi_1 \cdots \xi_k) = \xi_1 \cdots \xi_k +$ lower degree terms
2) sym is a vector space isomorphism
3) sym is G-equivariant.

Proof: exercise (for 2) use 1) & the PBW theorem). □

Corollary: $sym: S(\mathfrak{g})^G \xrightarrow{\sim} U(\mathfrak{g})^G$, vector space isomorphism.

And now we can explain how Casimir $C \in \mathcal{U}(\mathfrak{sl}_2)$ arises (and also why it's G-invariant for char $\mathbb{F} \neq 2$)

Exercise: Show that under $\mathbb{F}[\mathfrak{g}]^G \xrightarrow{\sim} S(\mathfrak{g})^G$, $F(x) = tr(x^2)$ goes to $\frac{1}{2}h^2 + 2fe$, which under sym: $S(\mathfrak{g})^G \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})^G$ goes to $\frac{1}{2}h^2 + h + 2fe = C$.

2.2) Completion of proof

The monomials $\prod_{j=1}^{N} f_{\beta_j}^{k_j} \prod_{i=1}^{n-1} h_i^{\ell_i} \prod_{j=1}^{N} e_{\beta_j}^{m_j}$ form bases in both $\mathcal{U}(\mathfrak{g}), S(\mathfrak{g})$. For $F \in S(\mathfrak{g})$ (resp. $F' \in \mathcal{U}(\mathfrak{g})$) write $res(F) \in S(\mathfrak{h})$ (resp. $res(F') \in \mathcal{U}(\mathfrak{h})$) for the sum of all monomials in F (resp. F') that only contain h's. Let $\iota: S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ be the isomorphism that is the id w.r.t. the bases (unlike sym, ι is not G-equivariant). Under the nat'l ident'n $\mathcal{U}(\mathfrak{h}) \xrightarrow{\sim} S(\mathfrak{h})$,

$$res(F) = res(\iota(F)) \qquad (1).$$

In Sec 1.2 of Lec 14 we've introduced a shifted W-action on $\mathfrak{h}^*$: $w \cdot \lambda = w(\lambda + \rho) - \rho$ and the shift map $\tau: \mathfrak{h}^* \to \mathfrak{h}^*$, $\lambda \mapsto \lambda + \rho$, which gives $\tau: \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)} \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^W$, $[\tau F](\lambda) := F(\lambda + \rho)$.

We've seen, Sec 1.2 of Lec 14, that $res: \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ restricts to an algebra homomorphism $HC_\bullet: \mathcal{U}(\mathfrak{g})^G = Z \longrightarrow \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$. We need to show that $z \mapsto HC_z$ is an isomorphism proving Thm in Sec 1.2 of Lec 14.

Proof: We have sym: $S(\mathfrak{g})^G \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})^G$ & $\tau: \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)} \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^W$. It's enough to show that

(*) $F \mapsto \tau \circ res \circ sym(F): S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W$ (of vector spaces)

We'll prove (*) by comparing this map to res, which is an isomorphism. Note that G preserves $S^d(\mathfrak{g})$ $\forall d \Rightarrow S(\mathfrak{g})^G = \bigoplus_d S^d(\mathfrak{g})^G$. By Sec 1, $res: S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W \Leftrightarrow [res(S^d(\mathfrak{g})^G) \subset S^d(\mathfrak{h})^W]$ $\overset{\sim}{res}: S^d(\mathfrak{g})^G \xrightarrow{\sim} S^d(\mathfrak{h})^W$, $\forall d$. So (*) will follow from

(**) $\forall d$, $F \in S^d(\mathfrak{g})^G \Rightarrow \deg(\tau \circ res \circ sym(F) - res(F)) < d$.

By (1) in Lemma in Sec 2.1, $sym(F) = \iota(F) + $ l.d.t. (terms of degree $<d$) $\Rightarrow$ $res \circ sym(F) = res \circ \iota(F) + $ l.d.t. $= [(1)] = res(F) + $ l.d.t. And τ preserves top deg. term. (**) follows. $\square$

3) Characters.

Notation: consider the group ring $\mathbb{Z}[\Lambda]$ of the weight lattice Λ. We write e^λ for the element of $\mathbb{Z}[\Lambda]$ corresponding to $\lambda \in \Lambda$. The Weyl group $W = S_n$ acts on $\mathfrak{h}^*$ preserving $\Lambda (= \{\sum \lambda_i \varepsilon_i \mid \lambda_i \in \mathbb{Z}\})$ and hence on $\mathbb{Z}[\Lambda]$: $we^\lambda = e^{w(\lambda)}$.

We also consider the completed version $\mathbb{Z}[[\Lambda]]$ consisting of all infinite linear combinations of e^λ, $\lambda \in \Lambda$.

Definition: Let M be a representation of $\mathfrak{g}$ w. $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ w. $\dim M_\lambda < \infty$. Below we will call such a representation a **weight module**.
The (formal) **character** of M, $ch\, M := \sum_{\lambda \in \Lambda} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[[\Lambda]]$.

Example: The Verma module $\Delta(\lambda)$ has weight basis $\prod_{j=1}^{N} f_{\beta_j}^{k_j} v_\lambda$ w. weights $\lambda - \sum_{j=1}^{N} k_j \beta_j$. So

$$\mathrm{ch}\,\Delta(\lambda) = \sum_{k_1,\dots,k_N \geq 0} e^{\lambda - \sum k_j\beta_j} = e^{\lambda} \prod_{j=1}^{N} (1 + e^{-\beta_j} + e^{-2\beta_j} + \dots) = e^{\lambda} \prod_{j=1}^{N} (1 - e^{-\beta_j})^{-1}$$

Our goal is to compute $\mathrm{ch}\, L(\lambda)$ for $\lambda \in \Lambda^+$. This will conclude our study of finite dimensional irreducible representations of $\mathfrak{sl}_n$. Recall, Sec. 1.2. of Lec 14, $\rho = \frac{1}{2} \sum$ positive roots $\Rightarrow \langle \rho, h_i \rangle = 1, \forall i$, so $\rho \in \Lambda$.

Thm (Weyl character formula): Let $\lambda \in \Lambda^+$. Then

$$\mathrm{ch}\, L(\lambda) = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \mathrm{sgn}(w) e^{w\rho}}$$

Examples: 1) $L(0)$ is the trivial representation, and we recover $\mathrm{ch}\,\mathrm{triv} = e^0 (=1)$.

2) Let $n=2$. Then $L(n) = M(n)$ & $\mathrm{ch}\, M(n) = [\dim M(n)_i = 1$ for $i = n, n-2, \dots, -n$ and 0, else$] = e^n + e^{n-2} + \dots + e^{-n} = \frac{e^{n+1} - e^{-(n+1)}}{e - e^{-1}}$. Since ρ is identified with $\langle \rho, h \rangle = 1$, this agrees with the theorem.

Exercise: For $\mathfrak{g}$-representations M, M' as in the definition above, and a finite dimensional $\mathfrak{g}$-representation V, we have

$$\mathrm{ch}(M \oplus M') = \mathrm{ch}(M) + \mathrm{ch}(M'), \quad \mathrm{ch}(V \otimes M) = \mathrm{ch}(V)\,\mathrm{ch}(M).$$

3) Complements.

3.1) Connection to characters for group representations.

Let V be a rational representation of $G=SL_n(\mathbb{F})$. We can consider its usual character $\chi_V(g)=tr_V(g)$. Then $\chi_V\in\mathbb{F}[G]^G$. We explain how to recover χ_V from $ch(V)$ – note that V is also $\mathfrak{g}$-repr'n.

First of all, let $T\subset G$ be the subgroup of diagonal matrices. From $\lambda\in\Lambda$ we can produce an algebraic group homomorphism e^λ $T\to\mathbb{F}^\times$: for $\lambda=\lambda_1\epsilon_1+\ldots+\lambda_n\epsilon_n$ w. $\lambda_i\in\mathbb{Z}$ and $t=diag(t_1,\ldots,t_n)$ we set $e^\lambda(t)=t_1^{\lambda_1}\ldots t_n^{\lambda_n}$ (this is well-defined: the λ_i's are defined up to a common summand, and $t_1t_2\ldots t_n=1$). Then $ch(V)=\chi_V|_T$ (exercise).

Now take $g\in G$. We can uniquely write it as $g=g_sg_u$, where g_s is a diagonalizable element, and g_u is unipotent – all eigenvalues are equal to 1. One can show that, $\forall\ F\in\mathbb{F}[G]^G$, we have $F(g)=F(g_s)$. And every g_s is conjugate to an element in T. So, we can recover χ_V from $\chi_V|_T$.

3.2) Schur polynomial.

In fact, $ch\,L(\lambda)$ (w. $\lambda\in\Lambda^+$) is essentially the Schur polynomial s_λ, see, e.g. Section 6.1 in [RT1], for definition. The meanings of λ in s_λ & $ch\,L(\lambda)$ are somewhat different though, so we will tweak $ch\,L(\lambda)$ a bit.

Namely, we can fix $\lambda_1,\ldots\lambda_n$ and extend the action of $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$ by making $1\in\mathfrak{gl}_n$ acting by $\lambda_1+\ldots+\lambda_n$. Then we can view $ch\,L(\lambda)$ as an element of the group algebra of $\mathbb{Z}^n$, which is the ring of Laurent polynomials $\mathbb{Z}[x_1^{\pm1},\ldots,x_n^{\pm1}]$. For $\mu\in\mathbb{Z}^n$ lets write x^μ for the corresponding monomial $x_1^{\mu_1} \cdots x_n^{\mu_n}$. Then we can write

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in W} \operatorname{sgn}(w)\, x^{w(\lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(w)\, x^{w\rho}},$$

where now we take $\rho = (n-1, n-2, \dots, 1, 0)$. In the denominator we have the Vandermonde determinant $\det(x_i^{j-1})_{i,j=1,\dots,n}$. And in the numerator we have $\det(x_i^{\lambda_i + j - 1})_{i,j=1,\dots n}$. So if λ is a partition we recover the determinantal definition of the Schur polynomial.

Exercise: By looking at $\operatorname{ch} \Lambda^k \mathbb{F}^n$, $\operatorname{ch} S^k(\mathbb{F}^n)$ recover the following equalities:

$s_{(1^k)} = e_k$, the kth symmetric polynomial

$s_{(k)} = m_k$, the complete symmetric polynomial.