Representations of algebraic groups & their Lie algebras, XI.

1) Chevalley restriction theorem.

2) Proof of HC isomorphism.

3) Complements.

1) In the last lecture we have stated the Harish-Chandra isomorphism theorem for the center \( Z = \mathfrak{u}(g)^G \) of \( \mathfrak{u}(g) \) \((G = SL_n)\), to be recalled below. The first step is to prove an analog of HC isomorphism in an easier setting. Consider the symmetric algebra \( S(g) \); \( G \) acts by automorphisms \( \rightarrow \) subalgebra \( S(g)^G \) of invariants. We'll establish an isomorphism \( S(g)^G \rightarrow S(Y)^W \), where \( W (= S_n) \) is the Weyl group. This is known as the Chevalley restriction theorem.

We start with an exercise (compare to the case of \( Y^G \) in Sec 1.3 of Lec 10). Consider the adjoint action of \( G \) on \( g \): \( g, x = gxg^{-1} \).

Exercise: Show that \((x, y) := \text{tr}(xy)\) defines a \( G \) invariant non-degenerate symmetric bilinear form.

This gives rise to a \( G \) equivariant isomorphism \( g \rightarrow g^*, x \mapsto \text{tr}(x \cdot) \), hence an algebra isomorphism \( S(g)^G \cong S(g^*)^G \equiv \mathbb{F}[g]^G \), where, explicitly \( \mathbb{F}[g]^G = \{ F \in \mathbb{F}[g] \mid F(x) = F(gxg^{-1}), \forall x \in g, g \in G \} \).

We have the restriction homomorphism \( \text{res} : \mathbb{F}[g] \rightarrow \mathbb{F}[Y], F \mapsto F|_Y \).

Example: Pick \( k \geq 1 \). Set \( F(x) = \text{tr}(x^k) \in \mathbb{F}[g] \). Then \( \text{res}(F) = p_k, k \)-th power symmetric polynomial, \( p_k(\text{diag}(x_1, \ldots, x_n)) = \sum_{i=1}^n x_i^k \).
The action of $S_n = W$ on $\mathfrak{g}$, its reflection representation, gives rise to $W \subset \mathfrak{F} [\mathfrak{g}]$ by algebra automorphisms. The algebra of invariants, $\mathfrak{F} [\mathfrak{g}]^W$, is $\mathfrak{F} [p_1, \ldots, p_n]$, compare to Sec 1.4 of Lec 14.

**Thm** ($S_n$-special case of the Chevalley restriction thm)

res: $\mathfrak{F} [\mathfrak{g}] \rightarrow \mathfrak{F} [\mathfrak{g}]$ restricts to $\mathfrak{F} [\mathfrak{g}]^c \rightarrow \mathfrak{F} [\mathfrak{g}]^W$.

**Proof:** We need to show:

1. res$(\mathfrak{F} [\mathfrak{g}]^c) \subset \mathfrak{F} [\mathfrak{g}]^W$
2. $\mathfrak{F} [\mathfrak{g}]^W = \text{res}(\mathfrak{F} [\mathfrak{g}]^c)$.
3. res$(\mathfrak{F} [\mathfrak{g}]^c)$ is injective.

(a): **Exercise:** Pick $w \in W$ and a "monomial matrix" $M_w \in SL_n (\mathfrak{F})$, $M_w = (m_{ij})$ w. m. $m_{ij} \neq 0$ only if $i = w(j)$. Then $M_w \text{diag} (x_1, \ldots, x_n) M_w^{-1} = \text{diag} (x_{w(1)}, \ldots, x_{w(n)})$.

Now take $F \in \mathfrak{F} [\mathfrak{g}]^c$. Of course, $F (\text{diag} (x_1, \ldots, x_n)) = F (M_w \text{diag} (x_1, \ldots, x_n) M_w^{-1}) = F (\text{diag} (x_{w(1)}, \ldots, x_{w(n)})) \Rightarrow \text{res}(F) \in \mathfrak{F} [\mathfrak{g}]^W$.

(b): $\mathfrak{F} [\mathfrak{g}]^W$ is generated by $p_1 \ldots p_n$. By the example above, $p_i = \text{res}(F_i)$ w. $F_i = \text{tr}(x^i) \in \mathfrak{F} [\mathfrak{g}]^c$. (6) follows.

(c): Let $F \in \text{ker}(\text{res}) \cap \mathfrak{F} [\mathfrak{g}]^c$. It vanishes on every diagonal $- \text{and } 6/c$ of $G$-invariance, on every diagonalizable matrix. Those form a Zariski dense subset in $\mathfrak{g}$. So $F = 0$. \hfill $\Box$
We can identify \( f \) w \( f^\star \) using the trace form as well. So we get the map \( \text{res}: S(g) \xrightarrow{\sim} S(g^\star) \xrightarrow{\text{res}} S(f^\star) \xrightarrow{\sim} S(f) \), an algebra homomorphism extended from \( g \xrightarrow{\sim} g^\star \xrightarrow{\text{res}} f^\star \xrightarrow{\sim} f \). \( x \in g \mapsto \text{res}(x) \) w \( \text{tr}(xy) = \text{tr}(\text{res}(x)y) \) \( \forall y \in f \). It follows that on the basis \( e, f, h \), we have \( \text{res}(h) = h \) & \( \text{res}(e) = \text{res}(f) = 0 \) (e.g. \( \text{tr}(e_y) = 0 \) \( \forall y \in f \)).

2) Proof of HC isomorphism
We'll deduce the Harish-Chandra isomorphism from the Chevalley thm.

2.) \( \text{sym}: S(g) \xrightarrow{\sim} U(g) \)
We want to compare \( Z = U(g)^G \) to \( S(g)^G \). We'll do so by constructing a \( G \)-linear vector space isomorphism \( S(g) \xrightarrow{\sim} U(g) \).
Consider a map \( g^k \rightarrow U(g)^k \), \((f_1, \ldots, f_k) \rightarrow \frac{1}{k!} \sum_{\sigma \in S_k} f_{\sigma(1)} \cdots f_{\sigma(k)} \). This map is multi-linear and symmetric (the image stays the same if we permute the arguments) so gives a unique linear map \( \text{sym}: S^k(g) \rightarrow U(g) \), w. \( f_1 \cdots f_k \rightarrow \frac{1}{k!} \sum_{\sigma \in S_k} f_{\sigma(1)} \cdots f_{\sigma(k)} \). Extend it to \( S(g) \) by linearity.

Lemma: The map \( \text{sym} \) has the following properties:
1) \( \text{sym}(f_1 \cdots f_k) = f_1 \cdots f_k + \text{lower degree terms} \)
2) \( \text{sym} \) is a vector space isomorphism
3) \( \text{sym} \) is \( G \)-equivariant.
Proof: exercise (for 2) use 1) & the PBW theorem). \( \square \)

Corollary: \( \text{sym}: S(g)^G \xrightarrow{\sim} U(g)^G \), vector space isomorphism.
And now we can explain how Casimir $C \in U(g)^c$ arises (and also why it's $G$-invariant for char $F \neq 2$)

**Exercise**: Show that under $F(x)^c \mapsto S(x)^c$, $F(x) = \text{tr}(x^3)$ goes to $\frac{1}{2}h^2 + 2fe$, which under sym: $S(x)^c \mapsto U(x)^c$ goes to $\frac{1}{2}h^2 + h + 2fe = C$.

### 2.2) Completion of proof

The monomials $\prod f_{j_i}^{b_i} \prod \hat{f}_{j_i}^{d_i} e_{b_i}^{e_i}$ form bases in both $U(g), S(g)$. For $F \in S(g)$ (resp. $F \in U(g)$) write $\text{res}(F) \in S(\hat{\mathfrak{g}})$ (resp. $\text{res}(F') \in U(\hat{\mathfrak{g}})$) for the sum of all monomials in $F$ (resp $F'$) that only contain $h$'s.

Let $\iota: S(g) \to U(g)$ be the isomorphism that is the id w.r.t the bases (unlike sym, $\iota$ is not $G$-equivariant). Under the naive identn $U(\hat{\mathfrak{g}}) \cong S(\hat{\mathfrak{g}})$,

$$\text{res}(F) = \text{res}(\iota(F)) \quad (1)$$

In Sec 1.2 of Lee 14 we've introduced a shifted $W$-action on $\mathfrak{g}^* : W : \lambda = w(\lambda + \rho) - \rho$ and the shift map $\tau : \mathfrak{g}^* \to \mathfrak{g}^*$, $\lambda \mapsto \lambda + \rho$ which gives $\tau : \mathcal{F}[\mathfrak{g}^*[W^*] \to \mathcal{F}[\mathfrak{g}^*][W]F(\lambda) = F(\lambda + \rho)$.

We've seen, Sec 1.2 of Lee 14, that $\text{res} : U(g) \to U(\hat{\mathfrak{g}})$ restricts to an algebra homomorphism $HC : U(g)^c = \mathbb{Z} \longrightarrow \mathcal{F}[\mathfrak{g}^*[W^*]$ We need to show that $\lambda \mapsto HC^2$ is an isomorphism proving Thm in Sec 1.2 of Lee 14.

**Proof**: We have $\text{sym} : S(g)^c \cong U(g)^c$ & $\tau : \mathcal{F}[\mathfrak{g}^*[W^*] \to \mathcal{F}[\mathfrak{g}^*[W]

It's enough to show that
\((*)\) \(F \mapsto \tau \circ \text{res} \circ \text{sym}(F) : S^d(g) \rightarrow S^d(f)^W\) (of vector spaces)

We'll prove \((*)\) by comparing this map to res, which is an isomorphism.

Note that \(G\) preserves \(S^d(g) \not\equiv d \rightarrow S^d(g) = \bigoplus S^d(g)\) \(G\). By Sec 1,

\[ \text{res} : S^d(g) \rightarrow S^d(f)^W \iff [\text{res}(S^d(g)) \subset S^d(f)^W] \text{ res} : S^d(g) \rightarrow S^d(f)^W \not\equiv d. \]

So \((*)\) will follow from

\((**)\) \(\forall d, F \in S^d(g) : \deg (\tau \circ \text{res} \circ \text{sym}(F) - \text{res}(F)) < d.\)

By \((1)\) in Lemma in Sec 2.1, \(\text{sym}(F) = \lambda(F) + \text{c.d.t. (terms of degree \(\leq d\) \(\rightarrow \)) }\)

\[ \text{res} \circ \text{sym}(F) = \text{res} \circ \lambda(F) + \text{c.d.t. } = [\lambda] = \text{res}(F) + \text{c.d.t.} \text{ And } \tau \text{ preserves top deg term. } (**) \text{ follows. } \]

\[ \square \]

3) Characters.

Notation: consider the group ring \(\mathbb{Z}[\Lambda]\) of the weight lattice \(\Lambda\). We write \(e^\lambda\) for the element of \(\mathbb{Z}[\Lambda]\) corresponding to \(\lambda \in \Lambda\). The Weyl group \(W = S_n\) acts on \(\mathbb{Z}^*\) preserving \(\Lambda(= \Sigma \lambda_i e_i; \lambda_i \in \mathbb{Z})\) and hence on \(\mathbb{Z}[\Lambda]\): \(we^\lambda = e^{w(\lambda)}\)

We also consider the completed version \(\mathbb{Z}[\Lambda]\) consisting of all infinite linear combinations of \(e^\lambda, \lambda \in \Lambda\).

Definition: Let \(M\) be a representation of \(g\) with \(M = \bigoplus_{\lambda \in \Lambda} M_\lambda, \dim M_\lambda < \infty\).

Below we will call such a representation a **weight module**.

The (formal) **character** of \(M\), \(\text{ch } M : = \sum_{\lambda \in \Lambda} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[\Lambda]\).
**Example:** The Verma module $\Delta(\lambda)$ has weight basis $\bigoplus_{k \in \mathbb{Z}} V_{\lambda+k}$. So

$$\operatorname{ch} \Delta(\lambda) = \sum_{k \equiv k_0} e^{\lambda - \sum k \beta_j} = e^{\lambda} \prod_{j=1}^{\infty} (1 + e^{-\beta_j} + e^{-2\beta_j} + \ldots) = e^{\lambda} \prod_{j=1}^{\infty} (1 - e^{-\beta_j})^{-1}$$

Our goal is to compute $\operatorname{ch} \mathcal{L}(\lambda)$ for $\lambda \in \Lambda^+$. This will conclude our study of finite dimensional irreducible representations of $\mathfrak{g}_0$. Recall, Sec. 1.2 of Lec 14, $p = \frac{1}{2} \sum$ positive roots $\Rightarrow \langle p, h_i \rangle = 1$, $\forall i$, so $p \in \Lambda$.

**Thm (Weyl character formula):** Let $\lambda \in \Lambda^+$. Then

$$\operatorname{ch} \mathcal{L}(\lambda) = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+p)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{wp}}$$

**Examples:**

1) $\mathcal{L}(0)$ is the trivial representation, and we recover $\operatorname{ch} \text{triv} = e^{0} (= 1)$.

2) Let $n=2$. Then $\mathcal{L}(\mathbf{1}) = M(\mathbf{1})$ & $\operatorname{ch} M(\mathbf{1}) = [\dim M(\mathbf{1})] \cdot 1$ for $i = n, n-1, \ldots, 0$, else $e^{\mathbf{n}+\mathbf{n}^{-1}+\ldots+\mathbf{n}} = \frac{e^{n+1} - e^{-n+1}}{e - e^{-1}}$. Since $p$ is identified with $\langle p, h \rangle = 1$, this agrees with the theorem.

**Exercise:** For $\mathfrak{g}$-representations $M, M'$ as in the definition above, and a finite dimensional $\mathfrak{g}$-representation $V$, we have

$$\operatorname{ch} (M \oplus M') = \operatorname{ch} (M) + \operatorname{ch} (M'), \quad \operatorname{ch} (V \otimes M) = \operatorname{ch} (V) \operatorname{ch} (M).$$
3) Complements.

3.1) Connection to characters for group representations.

Let \( V \) be a rational representation of \( G = \text{SL}_n(F) \). We can consider its usual character \( X_v(g) = tr_V(g) \). Then \( X_v \in IF[G] \). We explain how to recover \( X_v \) from \( \text{ch}(V) \)—note that \( V \) is also \( g \)-reprn.

First of all, let \( T < G \) be the subgroup of diagonal matrices. From \( \lambda \in \Lambda \) we can produce an algebraic group homomorphism \( e^\lambda \) \( T \rightarrow IF^\times \): for \( \lambda = \lambda_1 \xi_1 + \cdots + \lambda_n \xi_n \) w. \( \lambda_i \in \mathbb{Z}_+ \) and \( t = \text{diag}(t_1, \ldots, t_n) \) we set \( e^\lambda(t) = t_1^{\lambda_1} \cdots t_n^{\lambda_n} \) (this is well-defined: the \( \lambda_i \)'s are defined up to a common summand, and \( t_1 \cdots t_n = 1 \)). Then \( \text{ch}(V) = X_v |_T \) (Exercise).

Now take \( g \in G \). We can uniquely write it as \( g = g_sg_u \), where \( g_s \) is a diagonalizable element, and \( g_u \) is unipotent—all eigenvalues are equal to 1. One can show that, \( t \in IF[G] \), we have \( F(g) = F(g_s) \). And every \( g_s \) is conjugate to an element in \( T \). So, we can recover \( X_v \) from \( X_v |_T \).

3.2) Schur polynomial.

In fact, \( \text{ch} L(\lambda) \) (w. \( \lambda \in \Lambda^+ \)) is essentially the Schur polynomial \( S_\lambda \), see, e.g. Section 6.1 in [RT1], for definition. The meanings of \( \lambda \) in \( S_\lambda \) & \( \text{ch} L(\lambda) \) are somewhat different though, so we will tweak \( \text{ch} L(\lambda) \) a bit.

Namely, we can fix \( \lambda_1, \ldots, \lambda_n \) and extend the action of \( g_l^n \) to \( g_l^n \) by making \( t \in g_l^n \) acting by \( \lambda_1 + \cdots + \lambda_n \). Then we can view \( \text{ch} L(\lambda) \) as an element of the group algebra of \( \mathbb{Z}^n \), which is the ring of Laurent polynomials \( \mathbb{Z}[x_1^{-1}, \ldots, x_n^{-1}] \). For \( m \in \mathbb{Z}^n \) let's write \( x^m \) for the
corresponding monomial $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$. Then we can write

$$
\text{ch} \; \chi(\lambda) = \frac{\sum_{\text{w} \in \mathcal{W}} \text{sgn}(w) x^{w(\lambda + \mathbf{p})}}{\sum_{\text{w} \in \mathcal{W}} \text{sgn}(w) x^{w \mathbf{p}}},
$$

where now we take $\mathbf{p} = (n-1, n-2, \ldots, 1, 0)$. In the denominator we have the Vandermonde determinant $\det(x_i^{j-1})_{i,j=1}^{n}$. And in the numerator we have $\det(x_i^{\lambda_i + j-1})_{i,j=1}^{n}$. So if $\lambda$ is a partition we recover the determinantal definition of the Schur polynomial.

**Exercise:** By looking at $\text{ch} \; \Lambda^n \mathbb{F}^n$, $\text{ch} \; S^n(\mathbb{F}^n)$ recover the following equalities:

- $S_{(\lambda)} = e_k$, the $k$th symmetric polynomial
- $S_{(\mathbf{k})} = m_k$, the complete symmetric polynomial.