Representations of algebraic groups & Lie algebras, part XII.

1) Weyl character formula

2) What's next?

1) $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F})$ ($\mathbb{F}$ is alg. closed of char 0). A weight module is a $\mathfrak{g}$-representation M w. $M=\bigoplus_{\lambda\in\Lambda} M_\lambda$ w. $\dim M_\lambda<\infty$. In Sec 3 of Lec 15 we've defined the character $\operatorname{ch} M=\sum_{\lambda\in\Lambda}(\dim M_\lambda)e^\lambda$. For example,

$$\operatorname{ch}\Delta(\lambda)=e^\lambda\prod_{j=1}^N(1-e^{-\beta_j})^{-1}=e^{\lambda+\rho}/(e^\rho\prod(1-e^{-\beta_j})).$$

where $N=n(n-1)/2$, and $\beta_1,\dots,\beta_N$ are all positive roots. The goal of this part is to prove the following

Thm: for $\lambda\in\Lambda_+$, have $\operatorname{ch} L(\lambda)=\dfrac{\sum_{w\in W}\operatorname{sgn}(w)e^{w(\lambda+\rho)}}{\sum_{w\in W}\operatorname{sgn}(w)e^{w\rho}}$ ($W=S_n$, the Weyl grp).

The key idea is to express $\operatorname{ch} L(\lambda)$ via the characters of Verma modules.

1.1) JH filtration of a Verma module.

The first step here is the following.

Proposition: Let $\lambda\in\Lambda$. Then there is a $\mathfrak{g}$-module filtration $\{0\}=M_0\subset M_1\subset\ldots\subset M_{k+1}=\Delta(\lambda)$ s.t. $M_{k+1}/M_k\simeq L(\lambda)$ and, for $i\leqslant k$, $M_i/M_{i-1}\simeq L(\mu_i)$ w. $\mu_i\in W\cdot\lambda$ & $\mu_i<\lambda$ (i.e. $\lambda-\mu_i$ is the sum of positive roots).

Let's explain why we care.

Lemma: Let M be a weight module & $N \subset M$ be a submodule. Then

(1) $N, M/N$ are weight modules

(2) $\forall \lambda$, have a SES $0 \to N_\lambda \to M_\lambda \to M_\lambda/N_\lambda \to 0$.

(3) $\operatorname{ch} M = \operatorname{ch} N + \operatorname{ch} M/N$.

Proof: exercise.

Let $\lambda_1, \dots, \lambda_\ell$ be all elements of $W \cdot \lambda$ ordered in such a way that $\lambda_i \geq \lambda_j \Rightarrow i \leq j$ (e.g. λ_1 is the largest & λ_m is the smallest). Let's write m_{ij} for the multiplicity of $L(\lambda_j)$ in $\Delta(\lambda_i)$ (i.e. the number of occurrences of $L(\lambda_j)$ in a JH filtration of $\Delta(\lambda_i)$. By Proposition, the matrix (m_{ij}) is uni-triangular (= upper-triangular w. 1's on the diagonal). So (m_{ij}) is invertible, let (n_{ij}) denote the inverse, also uni-triangular.

Also, by Proposition, the only irreducibles that occur in $\Delta(\lambda_i)$ are $L(\lambda_j)$'s, $j > i$. So applying (3) of Lemma several times, we get

$$\operatorname{ch} \Delta(\lambda_i) = \operatorname{ch} L(\lambda_i) + \sum_{j>i} m_{ij} \operatorname{ch} L(\lambda_j)$$

$$\Updownarrow$$

$$\operatorname{ch} L(\lambda_i) = \operatorname{ch} \Delta(\lambda_i) + \sum_{j>i} n_{ij} \operatorname{ch} \Delta(\lambda_j). \qquad (1)$$

We'll deduce the theorem from (1) (& a few other things).

Proof of Proposition: Take a weight module M s.t.

(I) $\exists \tilde{\lambda} \in \Lambda \mid M_\mu \neq \{0\} \Rightarrow \mu \leq \tilde{\lambda}$ ("weights of M are bounded from above")

(II) $\exists \lambda' \in \Lambda$ s.t., $\forall z \in Z$ (the center), $HC_z(\lambda')$ is the unique eigenvalue for the action of z on M.

Claim: M admits a JH filtration by $L(\mu)$'s w. $\mu\in W\cdot\lambda$. Moreover, the multiplicity of $L(\mu)$ in M is $\leq \dim M_\mu$.

Exercise: deduce the proposition applying the claim to $\tilde{\lambda}=\lambda'=\lambda$, $M=\ker[\Delta(\lambda)\twoheadrightarrow L(\lambda)]$ (w. $M_{k-1}=M$).

Proof of claim: Suppose $M_0=\{0\}\subset M_1\subset\ldots\subset M_k=M$ is a filtration by $\mathfrak{g}$-subreps. By Lemma, each M_i/M_{i-1} is a weight module whose weights are $\leq\tilde{\lambda}$, by (I). So M_i/M_{i-1} has a highest weight, say μ_i w. the corresponding vector v_i. We have $xv_i=\langle\mu_i,x\rangle v_i$, $e_\alpha v_i=0$ $\forall\, x\in\mathfrak{h}$, positive root α $\exists!$ nonzero homomorphism $\Delta(\mu_i)\longrightarrow M_i/M_{i-1}$, $v_{\mu_i}\mapsto v_i$. But $z\in Z$ acts on $\Delta(\mu_i)$ by $HC_z(\mu_i)$. By (II), $HC_z(\mu_i)=HC_z(\lambda')\Leftrightarrow$ [Cor in Sec 1.2 of Lec 14] $\mu_i\in W\cdot\lambda'$.

Note that $\dim(M_i/M_{i-1})_{\mu_i}>0$. Since $\mu_i\in W\cdot\lambda'$, we get

$$k\leq\sum_{i=1}^{k}\dim(M_i/M_{i-1})_{\mu_i}=\sum_{i=1}^{k}\sum_{\mu\in W\cdot\lambda}\dim(M_i/M_{i-1})_\mu=[(2)\text{ of Lemma}]=\sum_{\mu\in W\cdot\lambda}\dim M_\mu.$$

This implies that M has a JH filtration (cannot refine a filtration by $\mathfrak{g}$-subreps indefinitely). It remains to prove that $L(\mu)$ occurs $\leq\dim M_\mu$ times: this follows from (2) of Lemma & $\dim L(\mu)_\mu=1$. $\square$

1.2) Proof of the Weyl character formula ($\lambda\in\Lambda^+$)

Step 1: By (1), for $w\in W$ $\exists\, n_w\in\mathbb{Z}$ w. $n_1=1$ s.t.

$$\operatorname{ch}L(\lambda)=\sum_{w\in W}n_w\operatorname{ch}\Delta(\overset{w(\lambda+\rho)-\rho}{w\cdot\lambda})=\Big(\sum_{w\in W}n_w e^{w(\lambda+\rho)}\Big)\Big/\Big(e^\rho\prod_{j=1}^{N}(1-e^{-\beta_j})\Big).\qquad(2)$$

Step 2: Recall $W \curvearrowright \mathbb{Z}[\Lambda]$, $we^{\lambda} = e^{w\lambda}$. We claim that $ch\, L(\lambda)$ is W-invariant. Indeed, in Sec 1.3 of Lec 13 (iii) of proof of Prop'n) we've seen that $\dim \tilde{L}(\lambda)_\mu = \dim \tilde{L}(\lambda)_{s_k\mu}$ $\forall\, k=1,\dots n-1$. And we know $\tilde{L}(\lambda) = L(\lambda)$, Corollary in Sec 1.3 of Lec 14 (alternatively, once we know $\dim L(\lambda) < \infty$, we can apply the argument of iii) to prove $\dim L(\lambda)_\mu = \dim L(\lambda)_{s_k\mu}$). Since s_k's generate W, we see that $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$ $\forall\, \mu \in \Lambda$, $w \in W$. Equivalently, $ch\, L(\lambda)$ is W-invariant.

Step 3: Step 2 invites a question: how do numerator/denominator of (2) behave under the action of W? We claim $e^{\rho}\prod_{j=1}^{N}(1-e^{-\beta_j})$ is <u>W-sgn-invariant</u>: applying $w \in W$ multiplies it by $sgn(w)$. It's enough to check this for $w = s_k$, where the claim follows from:

Exercise: Prove that $\prod_{j \mid \beta_j \neq \alpha_k}(1-e^{-\beta_j})$ is s_k-invariant, while $s_k(e^{\rho} - e^{\rho-\alpha_k})$ $[\langle \rho, h_k\rangle = 1] = e^{\rho-\alpha_k} - e^{\rho}$.

Step 4: The denominator of (2) is W-sgn-invariant (Step 3), while (2) is W-invariant (Step 2). So the numerator, $\sum_{w \in W} n_w e^{w(\lambda+\rho)}$ is W-sgn-invariant. But this means $n_w = sgn(w) n_1 = [n_1 = 1] = sgn(w)$.

Step 5: It remains to show $e^{\rho}\prod_{j=1}^{N}(1-e^{-\beta_j}) = \sum_{w\in W} sgn(w) e^{w\rho}$. For this apply (2) to $\lambda = 0$ getting 1 on the l.h.s. (side comment: this formula boils down to the factorization of the Vandermonde determinant, compare to the complement section of Lec 15.) $\square$

2) What's next?

One can ask how to compute $\operatorname{ch} L(\lambda)$ if λ is not dominant. In this case Step 2 fails (if λ is not dominant, take the unique dominant $\mu \in W\lambda$, then $\dim L(\lambda)_\lambda = 1 \neq \dim L(\lambda)_\mu = [\lambda < \mu] = 0$).

We still know that if $\lambda_1, \dots \lambda_\ell$ are all elements of $W\cdot\lambda$ written in the "decreasing" order as in Sec 1.1, then

$$\operatorname{ch} \Delta(\lambda_i) = \operatorname{ch} L(\lambda_i) + \sum_{j=i+1}^{\ell} m_{ij} \operatorname{ch} L(\lambda_j)$$

$$\Updownarrow$$

$$\operatorname{ch} L(\lambda_i) = \operatorname{ch} \Delta(\lambda_i) + \sum_{j=i+1}^{\ell} n_{ij} \operatorname{ch} \Delta(\lambda_j) \quad \text{w. } (n_{ij}) = (m_{ij})^{-1}$$

So our task is to compute the numbers n_{ij} (or m_{ij} - note that $m_{ij} \geqslant 0$) A conjectural answer was stated by Kazhdan and Lusztig in 1979 (a.k.a. Kazhdan-Lusztig, "KL" conjecture) and shortly thereafter proved independently by Beilinson & Bernstein and Brylinski & Kashiwara (1981) (using one more paper by Kazhdan-Lusztig from 1980).

These papers revolutionized the subject of Representation theory in (at least) two ways:

1) The form of the answer: the character formulas known before the KL conjecture were based purely on the classical enumerative combinatorics - compare to the Weyl character formula or the case of S_n-irreps (Sections 5.3, 6.1.2 of [RT1]). In contrast, the formula for m_{ij}'s (or n_{ij}'s) uses new objects - Kazhdan-Lusztig polynomials, elements

of $\mathbb{Z}_{\geq 0}[v]$ indexed by pairs $u, w \in W$. They are only defined recursively using the "Hecke algebra" $\mathcal{H}_v(W)$ and no enumerative description is known, in general — and none is expected to exist. The numbers m_{ij}, n_{ij} are then obtained as values of these polynomials at ± 1.

The importance of KL polynomials (and related structures) goes far beyond the representation theory of semisimple Lie algebras — essentially every known nontrivial character formula involves some version of these polynomials. And in the cases, where the character formulas are not known, we hope to relate the characters to KL polynomials (or their relatives).

Somewhat oversimplifying, one can say that KL polynomials in character formulas reflect the nontrivial categorical structure.

2) Methods of proof: the problem of computing $\mathrm{ch}\, L(\lambda)$ is purely algebraic/representation-theoretic. Nevertheless the approach in the aforementioned papers is geometric. The Kazhdan-Lusztig 1979 paper wasn't the first paper on Geometric representation theory (born around mid 70's to study the representations of finite groups of Lie type from Deligne's work on Weil's conjectures) but is a less "obvious" application and is one of the cornerstones in the field. Starting 1980's, Geometric representation theory became the central branch of the subject.

The relevant geometry is that of so called Schubert subvarieties (in flag varieties). These are very classical objects that date back to the 19th century enumerative geometry (see Fulton's "Young Tableaux..." for this classical story). We'll discuss some more about the Schubert varieties and the role they play in the proof of the KL conjecture in the end of the class.

Here's what we are going to discuss next. To finish our discussion of representation theory of algebraic groups and their Lie algebras we discuss two topics.

- presentation of $\mathfrak{sl}_n(\mathbb{F})$ by generators and relations. This will lead us to Kac-Moody algebras. The finite dimensional Kac-Moody algebras are exactly the (semi) simple Lie algebras. But there are also interesting infinite dimensional Kac-Moody algebras more notably, the affine ones.
- The representation theory of $SL_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F})$ for char $\mathbb{F}=p>0$. I plan to discuss some known results (we know a lot if $p\gg 0$) and mention open problems (p not so huge).

Then this class will have a "change of guard." While the Lie algebra $\mathfrak{sl}_n(\mathbb{F})$ (char $\mathbb{F}=0$) and it's representations (in category $\mathcal{O}$) are going to be featured very prominently, in some sense, it won't be the main player. The Hecke algebra/category will be. A preliminary plan for this part is as follows:

- The most elementary appearance of Hecke algebras in Representation theory is in the study of representations of finite groups of Lie type (such as $GL_n(\mathbb{F}_q)$). This is what we start with.

· Then we cover the generic Hecke algebra $\mathcal{H}_v(W)$ & its KL basis.

· The rest will have to do with the object called the "Hecke category" - one of the most important categories in Geometric representation theory with several different - but related - incarnations. We'll discuss Soergel's approach to this category / the proof of the KL conjecture based on studying the Soergel (bi)modules (1990), the most elementary incarnation of the Hecke category, which played a very important role in Representation theory, in the last decade, in particular.