Representations of algebraic groups & Lie algebras, part XIII.

1) Representations of $SL_n(\mathbb{F})$.

2) Complements.

Here $\mathbb{F}$ is an arbitrary algebraically closed field. Set $G=SL_n(\mathbb{F})$.

Goal: classify the irreducible rational representations of $G$.

When char $\mathbb{F}=0$, this has been already accomplished in Sec 1 of Lec 13 (Remark 2). What we'll do here works in char $p$ as well & for all semisimple groups. Our approach generalizes what was done for $SL_2(\mathbb{F})$ in Lec 11.

## 1.1) Weight decomposition.

We consider the "max'l torus" $T=\{diag(t_1,\dots,t_n)|t_1\dots t_n=1\}$. The following generalizes the $SL_2$-case (Lemma in Sec 1 of Lec 11).

Exercise: 1) Every rational representation of $T$ decomposes into the direct sum of 1-dimensional representations.

2) 1-dimensional rational representations of $T$ are in bijection with the weight lattice $\Lambda=\{\sum_{i=1}^n \lambda_i \varepsilon_i \mid \lambda_i\in\mathbb{Z}\}/Span_{\mathbb{Z}}(\varepsilon_1+\dots+\varepsilon_n)$ via $\lambda\in\Lambda\mapsto \mathbb{F}_\lambda$ w. action $t=diag(t_1,\dots,t_n)\mapsto \chi_\lambda(t):=t_1^{\lambda_1}\dots t_n^{\lambda_n}$ (well-defined b/c $t_1\dots t_n=1$).

Corollary: Let $V$ be a rational reprentation of $G$. It decomposes as $V=\bigoplus_{\lambda\in\Lambda} V_\lambda$, where $t\in T$ acts on $V_\lambda$ via $\chi_\lambda(t)$.

Recall that $\Lambda$ comes with an order: $\lambda\leqslant\mu$ if $\mu-\lambda$ is $\mathbb{Z}_{\geqslant 0}$-linear combination of positive roots $\varepsilon_i-\varepsilon_j$ $(i<j)$ $\rightsquigarrow$ highest and lowest weights of $V$.

**Theorem:** The irreducible rational representations of $G$ are in bijection with the set of dominant weights, $\Lambda_+$, via taking the highest weight. C1

We'll sketch a proof below.

We can also classify the irreps by their lowest weights. Let's explain how to recover them from highest ones. For this we will need some notation. Recall that $W$ is the Weyl group, $W=S_n$ acting on $\Lambda$ by permuting the entries. Consider $w_0\in W$: $w_0(i):=n+1-i$.

**Lemma:** Let $V$ be a rational representation of $G$. Then
(1) $V_\lambda \xrightarrow{\sim} V_{w\lambda}$ $\forall$ $w\in W$, $\lambda\in\Lambda$.
(2) If $\lambda$ is a highest weight of $V$, then $w_0\lambda$ is the lowest weight.

Proof: $W$ acts on $T$ as well (permutation of entries). For $t=$ $=\operatorname{diag}(t_1,\dots t_n)$, have $w.t=M_w t M_w^{-1}$, where $M_w\in G$ is a permutation matrix corresponding to $w$ ($M_w=(m_{ij})$ w. $m_{ij}\neq 0 \Leftrightarrow i=w(j)$, proof of (a) in Sec 1 of Lec 15). (1) follows from:

**Exercise:** The action of $\dot{M}_w$ on $V$ restricts to $V_\lambda \xrightarrow{\sim} V_{w\lambda}$.

To prove (2), note that $w_0$ sends the positive roots $\varepsilon_i - \varepsilon_j$ $(i<j)$ to negative roots and hence reverses the order on $\Lambda$. (2) follows $\square$

## 1.2) Sketch of proof of Thm:

It's in several steps:

Step 1: Let $B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$ be the subgroup of upper-triangular matrices, the "Borel subgroup." We have the projection $B \xrightarrow{\pi} T$ by taking the diagonal part. So we can view $\mathbb{F}_{w_0\lambda}$ as a representation of $B$. Let $\pi_{w_0\lambda} = \chi_{w_0\lambda} \circ \pi : B \to \mathbb{F}^\times$ be the corresponding homomorphism $\begin{pmatrix} t_1 & & * \\ & t_2 & \\ 0 & & \ddots\, t_n \end{pmatrix} \mapsto t_n^{\lambda_1} \dots t_1^{\lambda_n}$

For $\lambda \in \Lambda_+$ define the dual Weyl module:

$$M(\lambda) := \operatorname{Ind}_B^G \mathbb{F}_{w_0\lambda} = \{ f \in \mathbb{F}[G] \mid f(bg) = \pi_{w_0\lambda}(b) f(g),\ \forall b \in B,\ g \in G \},$$

where the $G$-action is given by $[gf](g') = f(g'g)$. See Sec 2 of Lec 11 for $SL_2$-case, there $M(\lambda) = \operatorname{Span}_{\mathbb{F}}(x^\lambda, x^{\lambda-1}y, \dots, y^\lambda)$. In general, we cannot describe $M(\lambda)$ so explicitly but we still have (a proof is in the complement section)

**Fact 1:** $\dim M(\lambda) < \infty$ & it's a rational $G$-representation.

The universal property of $M(\lambda)$ is

$$\operatorname{Hom}_G(V, M(\lambda)) \xrightarrow{\sim} \operatorname{Hom}_B(V, \mathbb{F}_{w_0\lambda}) \qquad (1)$$

Step 2: Let $V$ be a rational representation of $G$. Pick $\mu \in \Lambda$. Define $V_{\geqslant\mu} = \bigoplus_{\lambda \geqslant \mu} V_\lambda$. Define $V_{>\mu}$ similarly. For example, for $G = SL_2$, we have

$V_{\geqslant\mu} = \bigoplus_{n\geqslant 0} V_{\mu+2n}$. Similarly to page 6 of Lecture 11 notes, we have the following:

Fact 2: $V_{\geqslant\mu}, V_{>\mu}$ are $B$-stable, moreover $V_{\geqslant\mu}/V_{>\mu} \xrightarrow[B]{\sim} \mathbb{F}_\mu \otimes V_\mu$, where $V_\mu$ is the multiplicity space.

The proof is similar to the $SL_2$-case, see the complement section.

Step 3: We claim that $\dim M(\lambda)_\mu \neq 0 \Rightarrow \mu \leqslant \lambda$ ($\Leftrightarrow$ [Lemma in Sec 1.1] $\mu \geqslant w_0\lambda$) & $\dim M(\lambda)_\lambda = (\dim M(\lambda)_{w_0\lambda}) = 1$. For $SL_2$ this followed from the computation of $M(\lambda)$.

Fact 3: $\dim \operatorname{Hom}_B(\mathbb{F}_\lambda, M(\mu)) = \delta_{\lambda\mu}$.

This will also be proved in the complement section.

Now let $\lambda'$ be a highest weight of $M(\lambda)$. Then $M(\lambda)_{\geqslant\lambda'}$ is a $B$-submodule isomorphic to $\mathbb{F}_{\lambda'} \otimes \underbrace{M(\lambda)_{\lambda'}}$ ← multiplicity space. So

$$\operatorname{Hom}_B(\mathbb{F}_{\lambda'}, M(\lambda)) \hookleftarrow \operatorname{Hom}_B(\mathbb{F}_{\lambda'}, M(\lambda)_{\geqslant\lambda'}) = M(\lambda)_{\lambda'}$$

From Fact 3, we deduce that $\lambda' = \lambda$ (so $M(\lambda)_\mu \neq 0 \Rightarrow \mu \leqslant \lambda$) & $\dim M(\lambda)_\lambda = 1$.

Step 4: Now we can establish the existence of an irrep. w. highest weight $\lambda$. Consider the JH filtration $\{0\} = M_0 \subset M_1 \subset \ldots \subset M_k = M(\lambda)$. We have $(M_i/M_{i-1})_\lambda \neq 0$ for some (unique) $i$ and $\lambda$ is the highest weight of this module. So $M_i/M_{i-1}$ is the required irrep.

Step 5: Now we show the uniqueness. Let $L$ be an irreducible representation of $G$ w. highest weight $\lambda$ (and so lowest weight $-w_0\lambda$). As in the case of $SL_2$ we have

$$(L_{w_0\lambda})^* \simeq \mathrm{Hom}_B(L/L_{>w_0\lambda}, \mathbb{F}_{w_0\lambda}) \hookrightarrow \mathrm{Hom}_B(L, \mathbb{F}_{w_0\lambda}) \simeq \mathrm{Hom}_G(L, M(\lambda))$$

an iso, in fact, compare to Sol'n to Prob 5, HW2

So $L$ must embed into $M(\lambda)$. If we have two non-isomorphic $L, L'$, then repeating the argument for $SL_2$ (page 5 of Lec 11), we get $L \oplus L' \hookrightarrow M(\lambda)$. But then $L_\lambda \oplus L'_\lambda \hookrightarrow M(\lambda)_\lambda$. Since $L_\lambda, L'_\lambda \neq \{0\}$ but $M(\lambda)_\lambda = \mathbb{F}$ (Step 3), we arrive at a contradiction. $\square$

Define the Weyl module $W(\lambda) := M(-w_0\lambda)^*$. Note that $\mathrm{Hom}_G(W(\lambda), V) \xrightarrow{\sim} \mathrm{Hom}_B(\mathbb{F}_\lambda, V)$ (proof – exercise: use that $w_0^2 = 1$). Using this & Fact 3 we get

$$\dim \mathrm{Hom}_G(W(\lambda), M(\mu)) = \delta_{\lambda\mu} \qquad (2)$$

– compare to Prob. 5.3 in HW2.

Corollary (of proof):

1) $M(\lambda)$ has the unique irreducible subrepresentation, $L(\lambda)$; $L(\lambda)$ is also the unique irreducible quotient of $W(\lambda)$. C2.

2) Let $L$ be an irreducible rational representation of $G$. Then $\exists!\ \lambda \in \Lambda_+$ s.t. $\mathrm{Hom}_B(\mathbb{F}_\lambda, L) \neq 0$. This $\lambda$ is the highest weight of $L$. Moreover, $\dim \mathrm{Hom}_B(\mathbb{F}_\lambda, L) = 1$.

Proof: exercise.

## 1.3) Characters of irreducibles

**Lemma**: if char $\mathbb{F}=0$, then $W(\lambda) \xrightarrow{\sim} L(\lambda) \xrightarrow{\sim} M(\lambda)$

Proof: Recall (Sec 1.3 of Lec 14) that every finite dimensional $\mathfrak{g}$-representation is completely reducible. On the other hand,

$$\mathrm{Hom}_G(W(\lambda), M(\mu)) = [\text{Thm 2 in Sec 1.3 of Lec 7}] = \mathrm{Hom}_{\mathfrak{g}}(W(\lambda), M(\mu))$$

From (2) it follows that $W(\lambda)$ & $M(\mu)$ have $\delta_{\lambda\mu}$ common irreducible $\mathfrak{g}$-module direct summands.

**Exercise**: show that $L(\lambda)$ is irreducible over $\mathfrak{g}$ and deduce that $W(\lambda)$ & $M(\lambda)$ are irreducible. $\square$

To a rational $G$-representation $V$ we assign its character by the formula $\mathrm{ch}\, V = \bigoplus_{\lambda \in \Lambda} \dim V_\lambda \cdot e^\lambda$, compare to Sec 3 of Lec 15.

Lemma implies that $\mathrm{ch}\, M(\lambda) = \mathrm{ch}\, L(\lambda) = \mathrm{ch}\, W(\lambda)$ is given by the Weyl character formula (Thm in Sec 3 of Lec 15).

**Fact**: Over any $\mathbb{F}$, we have $\mathrm{ch}\, M(\lambda) = \mathrm{ch}\, W(\lambda)$ is given by the Weyl character formula.

The equality $\mathrm{ch}\, M(\lambda) = \mathrm{ch}\, W(\lambda)$ is an easy combinatorial observation. That $\mathrm{ch}\, M(\lambda)$ is given by the Weyl character formula follows from its geometric interpretation. Namely, the homogeneous space $G/B$ is the flag variety, $\mathcal{F}\ell$ of flags of subspaces

$\mathcal{F}=(\{0\}=V_0\subset V_1\subset\ldots\subset V_n=\mathbb{F}^n)$ w. $\dim V_i=i$. It's projective. Then $M(\lambda)$ is the space of global sections, $\Gamma(\mathcal{F}\ell,\mathcal{O}(\lambda))$ of a certain line bundle $\mathcal{O}(\lambda)$ on $\mathcal{F}\ell$. This already implies $\dim M(\lambda)<\infty$. Moreover, the higher cohomology $H^i(\mathcal{F}\ell,\mathcal{O}(\lambda))=0$ for $i>0$ – this is a special case of the Borel-Weil-Bott thm in char 0 & Kempf vanishing thm in char $p$. From the cohomology vanishing one can deduce that the character is independent of characteristic. References are in the complement section.

Now we proceed to the irreducible representations in char $p$. What we do below generalizes Section 2 in Lecture 10. Recall that we have the algebraic group homomorphism $Fr: G\to G$, $(x_{ij})\mapsto(x_{ij}^p)$.

For a representation $V$ of $G$ we can define its Frobenius twist $V^{(1)}$: if $\rho: G\to GL(V)$ is the homomorphism corresponding to $V$, then the homomorphism for $V^{(1)}$ is $\rho\circ Fr: G\to GL(V)$.

Exercise: $L(\lambda)^{(1)}\simeq L(p\lambda)$.

We have a complete analog of the Steinberg tensor product theorem, Corollary in Sec 2 of Lec 10. Define the set of "restricted" dominant weights $\Lambda_+^1:=\{\lambda=\lambda_1\varepsilon_1+\ldots+\lambda_n\varepsilon_n\in\Lambda_+ \mid \lambda_i-\lambda_{i+1}<p\ \forall\ i=1,\ldots,n-1\}$. We can then $p$-adically decompose an arbitrary element of $\Lambda$ as follows

Exercise: $\forall\ \lambda\in\Lambda_+\ \exists!\ k,\ \lambda_0,\ldots\lambda_k\in\Lambda_+^1$ (w. $\lambda_k\neq 0$) s.t. $\lambda=\sum_{i=0}^k p^i\lambda_i$.

**Thm** (Steinberg tensor product) For any $\lambda\in\Lambda^+$ & $\lambda_0,\dots,\lambda_k$ as above we have $L(\lambda)\simeq L(\lambda_0)\otimes L(\lambda_1)^{(1)}\otimes\dots\otimes L(\lambda_k)^{\underbrace{(k)}}$ ← k-fold Frobenius twist.

Moreover, $L(\lambda_i)$ is irreducible over $\mathfrak{g}$.

We don't prove this. Once we know that $L(\lambda')$ is irreducible over $\mathfrak{g}$ $\forall\,\lambda'\in\Lambda^1_+$, the proof works just as in the $SL_2$-case (Sec 2 of Lec 10), left as exercise. The irreducibility there follows from the explicit construction of $L(\lambda')$ ($\lambda'\in\{0,1,..,p-1\}$): $L(\lambda')=M(\lambda')$.

The latter equality is no longer true for general $n$ and no explicit construction of $L(\lambda')$ is known. The proof of the irreducibility in general is harder, the references are in the complement section.

The theorem allows us to reduce the computation of $ch\, L(\lambda)$ to the case $\lambda\in\Lambda^1_+$. The answer is known for $p\gg n$ and is a major open problem in the subject when $p$ is not so large, where a lot of progress has been achieved in the last 5 years or so. More details are in the complement section.

## 2) Complements: the separate note.