Proofs of Facts in Sec 1.2.

Proof of Fact 1: $\dim M(\lambda)<\infty$ & $M(\lambda)$ is a rational representation.

Recall $M(\lambda)=\{f\in\mathbb{F}[G]\mid f(bg)=\pi_{w_0\lambda}(b)f(g),\ \forall b\in B, g\in G\}$. The scheme of the proof is as follows:

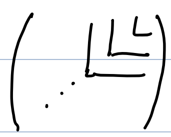
Step 1: We show that $M(\lambda)$ is the union of its finite dimensional rational subrepresentations.

Step 2: We show that $M(\lambda)_\mu\neq 0\Rightarrow \mu\leq\lambda$ & $\dim M(\lambda)_\mu<\infty$.

Step 3: We deduce the claim.

Step 1: In fact, $\mathbb{F}[G]$ itself (w. G-action coming from right translations) has this property (and then any subrepresentation does). An easy proof for $G=SL_n$ is as follows: note that the restriction map $\mathbb{F}[Mat_n]\to\mathbb{F}[SL_n]$ is G-equivariant & surjective. So it's enough to prove that $\mathbb{F}[Mat_n]$ is the union of finite dimensional rational subrepresentations. But the action of G on Mat_n from the right is a rational representation. And then $\mathbb{F}[Mat_n]=S(Mat_n^*)=\bigoplus_{i=0}^{\infty}S^i(Mat_n^*)$ and each $S^i(Mat_n^*)$ is a rational G-representation.

Step 2: uses essentially the same technique as in the computation of $M(\lambda)$ in Sec 2 of Lec 11: we restrict to a suitable Zariski open subset.

Let G^0 be the locus in G, where the $n-1$ anti-diagonal minors are nonzero: , i.e. $a_{1n}\neq 0$, $\det\begin{pmatrix}a_{1,n-1}&a_{1n}\\a_{2,n-1}&a_{2n}\end{pmatrix}\neq 0$, etc.

Let $\tilde{T}$ be the locus of antidiagonal matrices in SL_n: $\left\{\begin{pmatrix}0&&a_1\\&\cdot^{\cdot^{\cdot}}\ a_2&\\a_n&&0\end{pmatrix}\right\}$,

so that $M_{w_0}\in\tilde{T}$. Finally, let $U\subset B$ be the subgroup of all unitriangular matrices, $U=\{\begin{pmatrix}1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}\}$.

Exercise 1: (i) $B\times U\xrightarrow{\sim} G^\circ$ via $(b,u)\mapsto bM_{w_0}u$ for any choice of $M_{w_0}\in\tilde{T}$.

(ii) G° is $B\times B$-stable: $g\in G^\circ, b_1,b_2\in B \Rightarrow b_1gb_2\in G^\circ$

The restriction map $M(\lambda)\hookrightarrow N:=\{f\in\mathbb{F}[G^\circ] \mid f(bg)=\chi_{w_0\lambda}(b)f(g), \forall g\in G^\circ, b\in B\}$ is T-equivariant (for T acting by $[t.f](g)=f(gt)$).

So it's enough to show that $N=\bigoplus_{\mu\leq\lambda} N_\mu$ & $\dim N_\mu<\infty$ $\forall\mu$. Thanks to (i) of Exercise, we can identify N w. $\mathbb{F}[U]$ (via $f\mapsto f|_{M_{w_0}U}$).

Exercise 2: Under this identification, the action of T on N is given by $[t.f](u)=\chi_\lambda(t)f(t^{-1}ut)$ $(u\in U, t\in T, f\in\mathbb{F}[U])$.

U is the affine space w. coordinates x_{ij} that sends $u\in U$ to it's (i,j)-entry, here $i<j$. The weight of x_{ij} is $\varepsilon_j-\varepsilon_i$ (opposite of that of E_{ij}). So the weight of the monomial $\prod_{i<j} x_{ij}^{d_{ij}}\in\mathbb{F}[U]$ viewed as an element of N is $\lambda-\sum_{i<j} d_{ij}(\varepsilon_i-\varepsilon_j)\leq\lambda$. It also follows that $\dim N_\mu<\infty$, compare to ii) in Sec 1.3 of Lec 13.

This finishes the proof of Step 2.

Step 3: This is very similar to (iv) of Sec 1.3 of Lec 13. Namely, let $M\subset M(\lambda)$ be a rational subrepresentation. We have $M_\mu\neq 0\Rightarrow$ $\mu\leq\lambda$ and, since $\{\mu\in\Lambda \mid M_\mu\neq 0\}$ is W-stable (Lemma in Sec 1.1 of the lecture), $\mu\geq w_0\lambda$. Note that $\{\mu\in\Lambda \mid w_0\lambda\leq\mu\leq\lambda\}$ is finite. Also

$\dim M_\mu \leq \dim M(\lambda)_\mu \leq \dim N_\mu$. So, $\dim M \leq \sum_{w_0\lambda \leq \mu \leq \lambda} N_\mu$. Together with Step 1, this shows (exercise) that $\dim M(\lambda) < \infty$ and completes the proof.

Proof of Fact 2: $V_{\geq\mu}, V_{>\mu}$ are B-stable, and $V_{\geq\mu}/V_{>\mu} \xrightarrow{\sim} \mathbb{F}_\mu \otimes V_\mu$.

This follows from (**) in Sec 3 of Lec 11. Namely, the claim boils down to the claim that for $\forall \mu \in \Lambda$ & $v \in V_\mu$ we have $Uv \subset v + V_{>\mu}$. For a positive root $\alpha = \varepsilon_i - \varepsilon_j$ consider the subgroups

$$U_\alpha = \{1 + tE_{ij} \mid t \in \mathbb{F}\} \subset G_\alpha = \{aE_{ii} + bE_{ij} + cE_{ji} + dE_{jj} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\} \subset G$$

Applying (**) to $G_\alpha \simeq SL_2$, we see that $U_\alpha v \subset v + \sum_{n>0} V_{\mu + n\alpha}$. But notice that the subgroups U_α for all $\alpha = \varepsilon_i - \varepsilon_j$ $(i<j)$ generate U. $Uv \subset u + V_{>\mu}$ follows.

Proof of Fact 3: Essentially follows already from our proof of Fact 1. Namely, note that for a rational G-representation V, have

$$\mathrm{Hom}_B(\mathbb{F}_\lambda, V) \xrightarrow{\sim} \{v \in V_\lambda \mid Uv = v\} = V_\lambda \cap V^U$$

Recall the embedding $M(\mu) \hookrightarrow N$ in Step 2 of Proof of Fact 1. So $M(\mu)^U \hookrightarrow N^U$. Recall that we have identified N with $\mathbb{F}[U]$. The identification is U-equivariant (by (i) of Exercise 1 in that proof), where the action of U on $\mathbb{F}[U]$ comes from the action by left translations. The only U-invariant elements in $\mathbb{F}[U]$ are scalars. We have seen that their weight for the T-action is λ. So $N^U \subset N_\lambda$ & is 1-dim'l. Hence $\dim \mathrm{Hom}_B(\mathbb{F}_\lambda, M(\mu)) \leq \delta_{\lambda\mu}$. On the other hand as we've seen in Step 3 of the proof in the lecture, if λ' is a highest weight of $M(\mu)$, then $\mathrm{Hom}_B(\mathbb{F}_{\lambda'}, M(\mu)) \neq 0$. We conclude that $\dim \mathrm{Hom}_B(\mathbb{F}_\lambda, M(\mu)) = \delta_{\lambda\mu}$.