Hecke algebra/category, part 1.

1) Introduction.
2) $\mathrm{End}_G(\mathbb{C}[G/H])$ & convolution
3) The case of $B\subset G=GL_n(\mathbb{F}_q)$

1) In the 1st lecture of this class we have mentioned that in the study of representations of finite groups, G, we prioritize the case when G is (almost) simple. Most of such are finite groups of Lie type, the simplest example is $GL_n(\mathbb{F}_q)$, where q is a prime power. It is close to being simple: $PSL_n(\mathbb{F}_q):=SL_n(\mathbb{F}_q)/\text{center}$ is simple if $(n,q)\neq(2,2),(2,3)$. For a more detailed discussion of finite simple groups of Lie type see [St], Ch. 4.

We will concentrate on $\mathbb{C}GL_n(\mathbb{F}_q)$-modules (compare to S_n vs $\mathfrak{A}_n$, although passing from GL_n to PSL_n is way more complicated...)

We are not going to study all $GL_n(\mathbb{F}_q)$-irreps. Set $G=GL_n(\mathbb{F}_q)$ and let B be the subgroup of all upper triangular matrices.

Definition/lemma: for a $\mathbb{C}G$-irrep V, TFAE

(a) $V^B\neq 0$

(b) V appears as a direct summand in $\mathbb{C}[G/B]$ $(=\{f\in\mathbb{C}[G]\mid f(g'b)=f(g'),\ \forall g'\in G, b\in B\}$ w. G-action via $[g.f](g')=f(g^{-1}g'))$

We say that V is a unipotent principal series representation.

Proof: Observe that $\mathbb{C}[G/B]=\mathrm{Ind}_B^G\mathrm{triv}$ & use Frobenius reciprocity $\mathrm{Hom}_G(\mathbb{C}[G/B],V)=\mathrm{Hom}_B(\mathrm{triv},V)=V^B$ $\square$

Rem: 1) Recall that for rational representations (of $SL_n(\mathbb{F})$ - not much difference) there's a unique irreducible w. a B-fixed vector (2) of Corollary in Sec 1.2 of Lec 17) - the trivial one. The answer in our case turns out to be very different.

2) Unipotent G-irreps are interesting e.g. b/c the study of general irreps reduces to that of unipotent ones. In a way, they are analogs of unipotent conjugacy characters. [C] provides a comprehensive treatment of the representation theory of finite groups of Lie type.

One can parameterize principal series unipotent irreps as follows:

Lemma: There is a bijection

$$\begin{array}{ccc} \{\text{princ. series unipotent } G\text{-irreps}\} & \ni & V \\ \updownarrow\sim & & \downarrow \\ \{\text{irreducible right } \operatorname{End}_G(\mathbb{C}[G/B])\text{-modules}\} & \ni & \underbrace{M_V}_{\text{multiplicity space of } V \text{ in } \mathbb{C}[G/B].} := \operatorname{Hom}_G(\mathbb{C}[G/B], V) \end{array}$$

Proof: This is a basic fact about representations of semisimple assoc've algebras. It follows, for example, from [RT1], Lemma 2.3 (applied to $A = \operatorname{End}_{\mathbb{C}}(\mathbb{C}[G/B])$, $B = \mathbb{C}G$ so that $\operatorname{End}_G(\mathbb{C}[G/B])$ is the centralizer algebra $\square$

Our short term (this & the next lecture) goals are

1) Describe a basis in $\operatorname{End}_G(\mathbb{C}[G/B])$

2) Explain how the basis elements multiply recovering the algebra structure

3) Use this to identify $\operatorname{End}_G(\mathbb{C}[G/B])$ w. $\mathbb{C}S_n$ whose irreps we know.

2) $\mathrm{End}_G(\mathbb{C}[G/H])$ & convolution.

Let $H \subset G$ be finite groups $\rightsquigarrow$ $\mathbb{C}G$-module $\mathbb{C}[G/H] \rightsquigarrow$ algebra $\mathrm{End}_G(\mathbb{C}[G/H])$. By Frobenius reciprocity, have a vector space identification

double cosets

$$\mathrm{End}_G(\mathbb{C}[G/H]) = \mathrm{Hom}_G(\mathbb{C}[G/H], \mathbb{C}[G/H]) = \mathbb{C}[G/H]^H (= \mathbb{C}[G]^{H\times H} = \mathbb{C}[H\backslash G/H])$$

For $\mathcal{O} \in H\backslash G/H$, define $\delta_{\mathcal{O}} \in \mathbb{C}[H\backslash G/H]$ by $\delta_{\mathcal{O}}(\mathcal{O}') = \delta_{\mathcal{O},\mathcal{O}'}$. $\delta_{\mathcal{O}}$'s form a basis in $\mathbb{C}[H\backslash G/H] = \mathrm{End}_G(\mathbb{C}[G/H])$ accomplishing Goal 1 above (modulo an explicit parametrization of $H\backslash G/H$ in our case).

Now we proceed to Goal 2: we introduce the convolution product on $\mathbb{C}[H\backslash G/H]$.

Definition/lemma: Let $H_1, H_2, H_3 \subset G$ be subgroups. Then the convolution

$$*: \mathbb{C}[H_1\backslash G/H_2] \times \mathbb{C}[H_2\backslash G/H_3] \longrightarrow \mathbb{C}[H_1\backslash G/H_3] (= \mathbb{C}[G]^{H_1\times H_3}),$$

$f_1 * f_2(g) = \frac{1}{|H_2|} \sum_{g_1, g_2 \in G \mid g_1 g_2 = g} f_1(g_1) f_2(g_2)$, is well-defined (meaning that $f_1 * f_2$ is left H_1-invariant & right H_3-invariant). Left as exercise.

Example: Suppose $H_1 = H_2 = H_3 = \{1\}$. For $g_1, g_2 \in G$ consider $f_{12} = \delta_{g_1}$, $f_{23} = \delta_{g_2}$. Then $\delta_{g_1} * \delta_{g_2}(g) = \sum_{g_1', g_2' \mid g_1' g_2' = g} \delta_{g_1}(g_1')\delta_{g_2}(g_2') = \delta_{g_1 g_2}$. So $(\mathbb{C}[G], *)$ is identified with $\mathbb{C}G$.

Exercise: 1) In the general situation, $*$ is associative. In particular, $\mathbb{C}[H\backslash G/H]$ is an associative algebra & $\mathbb{C}[G/H]$ is a $\mathbb{C}[G]$-$\mathbb{C}[H\backslash G/H]$-bimodule.

2) For $\mathcal{O}_1 \in H_1\backslash G/H_2$, $\mathcal{O}_2 \in H_2\backslash G/H_3$ we have $\delta_{\mathcal{O}_1} * \delta_{\mathcal{O}_2} = \sum_{\mathcal{O}\in H_1\backslash G/H_3} m^{\mathcal{O}}_{\mathcal{O}_1,\mathcal{O}_2}\delta_{\mathcal{O}}$ where $m^{\mathcal{O}}_{\mathcal{O}_1,\mathcal{O}_2} = \frac{1}{|H_2||\mathcal{O}|}|\{g_1\in\mathcal{O}_1, g_2\in\mathcal{O}_2 \mid g_1g_2\in\mathcal{O}\}| \ (\in \mathbb{Z}_{\geqslant 0})$

3) In particular, if $H_1=H_2$, then $\delta_{H_1} * \delta_{\mathcal{O}} = \delta_{\mathcal{O}}$, and if $H_2=H_3$, then $\delta_{\mathcal{O}} * \delta_{H_2} = \delta_{\mathcal{O}}$. So δ_H is a unit in $\mathbb{C}[H\backslash G/H]$.

Corollary: 1) The action of $(\mathbb{C}[G],*)$ on $\mathbb{C}[G/H]$ coincides w. the representation of $\mathbb{C}G$ there. C3

2) The right action of $(\mathbb{C}[H\backslash G/H],*)$ on $\mathbb{C}[G/H]$ identifies $(\mathbb{C}[H\backslash G/H],*)$ w. $\mathrm{End}_G(\mathbb{C}[G/H])^{opp}$ ("opp" means opposite product).

Proof: 1) follows from $\delta_{g'} * \delta_{gH} = \delta_{g'gH}$. To prove 2) observe that 1) of Exercise gives a right action of the (unital) algebra $\mathbb{C}[H\backslash G/H]$ on $\mathbb{C}[G/H]$ commuting w. $\mathbb{C}G$, hence an algebra homomorphism

$$\mathbb{C}[H\backslash G/H] \longrightarrow \mathrm{End}_G(\mathbb{C}[G/H])^{opp} \qquad (1)$$

It's injective: for $\delta_H \in \mathbb{C}[G/H]$, have $\delta_H * f = f$, $\forall f\in\mathbb{C}[H\backslash G/H]$

Since the source and the target of (1) are isomorphic finite dimensional spaces, (1) is an algebra isomorphism $\square$

Remark: $\mathrm{End}_G(\mathbb{C}[G/H])^{opp} = \mathbb{C}[H\backslash G/H]$ is often referred to as the Hecke algebra of $H\subset G$.

3) The case of $B\subset G = GL_n(\mathbb{F}_q)$

Now we apply results of Section 2 to the case of interest.

3.1) Description of orbits.

Here we explain two results that are consequences of the Gauss elimination algorithm. Let $G = GL_n(\mathbb{F}) \supset B$ – the upper triangular matrices.

Let $\mathbb{F}$ be a field. Set $W = S_n$. For $w \in W$ let $M_w = (\delta_{i,w(j)})_{i,j=1}^n$ be the corresponding permutation matrix in G. Write BwB for $BM_wB \subset G$.

Fact 1 (Bruhat decomposition): $G = \coprod_{w\in W} BwB$

As a corollary, $G/B = \coprod_{w\in W} BwB/B$. The subsets $BwB \subset G$ are known as Bruhat cells, while $BwB/B \subset G/B$ are Schubert cells. Our goal now is to describe them more explicitly as sets. We need some terminology.

By the leghth, $\ell(w)$ of $w\in W$ we mean $|\{(i,j)\in\{1,\dots,n\}^2 \mid i<j, w(i)>w(j)\}|$ (a.k.a. the number of inversions). Alternatively, $\ell(w)$ can be described in the following way. The elements $s_i = (i,i+1), i=1,\dots,n-1$, generate W. Then $\ell(w)$ = the minimal length of a word in the s_i's equal to w (exercise).

Consider the subset $U_w \subset U = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}\right\}$ consisting of all matrices whose (i,j) entry is zero if $i<j$ & $w(i)<w(j)$. Clearly, $U_w \xrightarrow{\sim} \mathbb{F}^{\ell(w)}$ (via taking the potentially nonzero entries)

Fact 2: The map $U_w \times B \to BwB$, $(u,b)\mapsto uM_wb$ is a bijection. Hence $\mathbb{F}^{\ell(w)} \xrightarrow{\sim} BwB/B$.

3.2) Hecke algebra.

Now take $\mathbb{F}=\mathbb{F}_q$. Let $H(q)=(\mathbb{C}[B\backslash G/B],*)$. By Fact 1, it has basis $T_w=\delta_{BwB}$, $w\in W$. Note that $T_1=1$. Now we compute some products of T_w's.

Proposition: 1) if $\ell(uw)=\ell(u)+\ell(w)$, then $T_uT_w=T_{uw}$.

2) For $s=s_i$ $(i=1,\dots,n-1)$, we have $T_s^2=(q-1)T_s+qT_1$.

Proof: Consider the map $BuB\times BwB\xrightarrow{\pi}G$, $(x,y)\mapsto xy$. The group B acts on $BuB\times BwB$ by $b.(x,y)=(xb^{-1},by)$. This action is free & each fiber of π is a union of orbits. By 2) of Exercise in Sec 2,

(*) $T_uT_w=\sum_{v\in W} m_{uw}^v T_v$, where $m_{uw}^v=\#$ of B-orbits in $\pi^{-1}(z)$, $z\in BvB$.

1): Note that $BuwB\subset \operatorname{im}\pi$. By Fact 2, $|BuwB|=q^{\ell(uw)}|B|$, $|BuB\times BwB|=$
$=|B|^2q^{\ell(u)}q^{\ell(w)}=|B|^2q^{\ell(uw)}=|B|\cdot|BuwB|$. Since each fiber of π is a union of free B-orbits, we get $BuwB=\operatorname{im}\pi$ and each fiber is a single B-orbit. Our claim follows from (*).

2): Consider the subgroup $P_s=\begin{array}{cc} \overset{i}{\downarrow} & \\ \begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ 0 & & \ddots & * \end{pmatrix} & \leftarrow i+1 \end{array}$ so that $P_s=BsB\sqcup B$, (exercise). C4

By (*), $T_s^2=m_{ss}^sT_s+m_{ss}^1T_1$. First of all, $m_{ss}^1=\frac{1}{|B|}|\pi^{-1}(1)|=[\pi^{-1}(1)=(g,g^{-1})$, $g\Leftrightarrow g^{-1}\in BsB]=\frac{1}{|B|}|BsB|=q$. Next,

$|BsB\times BsB|=|\pi^{-1}(BsB)|+|\pi^{-1}(B)|=m_{ss}^s|BsB||B|+m_{ss}^1|B||B| \Rightarrow$ divide by $|B|^2$

$q^2=m_{ss}^sq+q\Rightarrow m_{ss}^s=q-1$. $\square$