Hecke algebra/category, part II.

1) Generic Hecke algebra
2) Generalizations: Kac-Moody algebras.
3) Complements.

1.0) Recap: Let's recall some results from Lecture 18.

Let $G=GL_n(\mathbb{F}_q)$, $B\subset G$ be the subgroup of upper triangular matrices. We are interested in understanding the algebra $\mathrm{End}_G(\mathbb{C}[G/B])$. It's semisimple ($\simeq \bigoplus$ matrix algebras) b/c $\mathbb{C}[G/B]$ is a completely reducible $\mathbb{C}G$-module.

In Sec 2 of Lec 18, we have identified $\mathrm{End}_G(\mathbb{C}[G/B])$ w. the algebra $\mathcal{H}(q):=(\mathbb{C}[B\backslash G/B],*)$. Using this, in Sec 3, we have produced a vector space basis $T_w\in \mathrm{End}_G(\mathbb{C}[G/B])$, $w\in W(:=S_n)$, where $T_1=1$.

We have also described the products of some basis elements. Recall that W is generated by $s_i:=(i,i+1)$, $i=1,\dots,n-1$. For $w\in W$ we defined its length $\ell(w):=\min\{\ell \mid w=s_{i_1}\dots s_{i_\ell}\}$, e.g. $\ell(1)=0$, $\ell(w)=1 \Leftrightarrow w=s_i$. The following was established in Sec 3 of Lec 18:

Proposition: 1) if $\ell(uw)=\ell(u)+\ell(w)$, then $T_uT_w=T_{uw}$.

2) For $s=s_i$ $(i=1,\dots,n-1)$, we have $T_s^2=(q-1)T_s+qT_1$.

1.1) Consequences.

Corollary: 1) The elements $T_i:=T_{s_i}$ $(i=1,\dots,n-1)$ generate $\mathcal{H}(q)$.

2) We have $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$ & $T_iT_j=T_jT_i$ for $|i-j|>1$

3) $T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) = \ell(w)+1 \\ q T_{sw} + (q-1) T_w, & \text{else} \end{cases}$

4) $T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) = \ell(w)+1 \\ q T_{ws} + (q-1) T_w, & \text{else} \end{cases}$

5) $\exists\ m_{uw}^v \in \mathbb{Z}[t]$ $(u,v,w \in W)$ s.t. $T_u T_w = \sum_{v \in W} m_{uw}^v(q) T_v$.

Proof: 1): Suppose $w = s_{i_1} \cdots s_{i_\ell}$ w. $\ell = \ell(w)$. Note that $\ell(su) \leq \ell(u)+1$ $\forall\, u \in W$
$\Rightarrow \ell(s_{i_k} \cdots s_{i_\ell}) = \ell - k + 1$ $\forall\, k = 1, \dots \ell$. By 1) of Proposition,

$$T_w = T_{i_1} \cdots T_{i_\ell} \tag{$*$}$$

2): from $s_i s_{i+1} s_i = (i, i+2) = s_{i+1} s_i s_{i+1}$ (length 3), $s_i s_j = s_j s_i$ (length 2).

3): The case $\ell(sw) = \ell(w)+1$ follows from 1) of Proposition. $\ell(sw) \leq \ell(w)+1$
$\Rightarrow \ell(w) = \ell(s^2 w) \leq \ell(sw)+1$. Since $sgn(w) = (-1)^{\ell(w)}$ & $sgn(sw) = -sgn(w)$, we get $\ell(sw) = \ell(w) \pm 1$. We only need to consider the case $\ell(w) = \ell(sw)+1$
$\Rightarrow T_w = T_s T_{sw}$. So $T_s T_w = T_s^2 T_{sw} =$ [2) of Prop'n: $T_s^2 = (q + (q-1) T_s)$]
$= q T_{sw} + (q-1) T_s T_{sw} =$ [1) of Prop'n] $q T_{sw} + (q-1) T_w$.
4): is similar.

5): We write u as $s_{i_1} \cdots s_{i_\ell}$ w. $\ell = \ell(u)$ so that $T_u T_w = [(*)] = T_{i_1} \cdots T_{i_\ell} T_w$. We use 3) repeatedly: express $T_{i_\ell} T_w$, then multiply the summands by $T_{i_{\ell-1}}$, etc. In each step, the coefficients of T_v's are polynomials in q with integral coefficients. $\square$

1.2) The generic Hecke algebra and its specializations.

Definition: The generic Hecke algebra (a.k.a. Iwahori-Hecke algebra) is the free $\mathbb{Z}[t]$-module $H^{\mathbb{Z}}(W)$ w. basis $T_w, w \in W$, and product

$$T_u T_w = \sum_{v \in W} \underbrace{m_{uw}^v(t)}_{\text{from 5) of Corollary}} T_v.$$

Lemma: This is an associative algebra w. unit T_1.

Proof: Associativity can be checked on basis elements, where it's a collection of quadratic equations on the entries of the multiplication table – $m_{uw}^v \in \mathbb{Z}[t]$. These equations hold after specializing t to <u>any</u> prime power q, by 5) of Corollary. So they hold for the m_{uw}^v, hence $H^{\mathbb{Z}}(W)$ is associative. The claim that T_1 is a unit is an exercise. $\square$

We write $\mathcal{H}(W)$ for $\mathbb{C} \otimes_{\mathbb{Z}} H^{\mathbb{Z}}(W)$. For $\kappa \in \mathbb{C}$, we write $\mathcal{H}_\kappa(W)$ for $\mathcal{H}(W)/(t-\kappa)\mathcal{H}(W)$. This is a $\mathbb{C}$-algebra w. basis $T_w, w \in W$ and product $T_u T_w = \sum_{v \in W} m_{uw}^v(\kappa) T_v$.

Example: 1) For $\kappa = q$, a prime power, $\mathcal{H}_q(W) = H(q)$, a semisimple algebra.

2) Let $\kappa = 1$. By 3) of Corollary, $T_s T_w = T_{sw} \Rightarrow T_u T_w = T_{uw}, \forall u, w \in W$.
$\Rightarrow \mathcal{H}_1(W) = \mathbb{C}W$.

It turns out that 1) & 2) already imply $\mathcal{H}_q(W) \simeq \mathcal{H}_1(W) = \mathbb{C}W$.

Theorem (Tits deformation principle): Let $\mathbb{F}$ be an algebraically closed field and A be an associative unital $\mathbb{F}[t]$-algebra that is a free finite rank $\mathbb{F}[t]$-module. Let $\alpha, \beta \in \mathbb{F}$ be such that A_α, A_β are semisimple. Then $A_\alpha \simeq A_\beta$.

Corollary: $\mathcal{H}_q(W) \simeq \mathbb{C}W$.

This accomplishes the last goal stated in the previous lecture and finishes our treatment of the representations theory of $GL_n(\mathbb{F}_q)$. Many more details are in [C].

Remarks: 1) We'll sketch the proof of the theorem in the complement section. Two key steps: to prove that $\mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}[t]} A \simeq \mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}} A_\alpha$ ($\mathbb{F}[t] \hookrightarrow \mathbb{F}[[t-\alpha]]$ via the expansion in $t-\alpha$) by "lifting of idempotents" and then use a bit of Algebraic geometry to finish the proof.

2) $\mathcal{H}_k(W)$ is semisimple $\Leftrightarrow k$ is not a root of unity of order $\leq n$. When k is a pth root of unity ($p \leq n$ is prime) than the representation theory of $\mathcal{H}_k(W)$ resembles that of $\overline{\mathbb{F}}_p W$ (but is much (!) easier).

3) One can ask to construct an isomorphism $\mathcal{H}_k(W) \xrightarrow{\sim} \mathbb{C}W$ explicitly. It's possible to construct an isomorphism with the third algebra, a "cyclotomic KLR (Khovanov-Lauda-Rouquier) algebra" that arises in the study of representations of Lie algebras in categories. See Kleshchev, arXiv: 0909.4844 for details.

2) Generalizations.

We've been looking at the Lie algebra $\mathfrak{sl}_n$, the algebraic group SL_n (or GL_n) and the Weyl group $W=S_n$. But various constructions and results we've seen generalize to semisimple or more general "Kac-Moody" Lie algebras (or groups) and their Weyl groups - or more general Coxeter groups. We will briefly review these objects starting with the Kac-Moody Lie algebras. And our starting point here is the presentation of $\mathfrak{sl}_n$ by generators & relations.

2.1) $\mathfrak{sl}_n(\mathbb{C})$ by generators & relations.

Notation: $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{C})$. For $i=1,\dots,n-1$ set $h_i=E_{ii}-E_{i+1,i+1}$ (form basis in $\mathfrak{h}$), $e_i=E_{i,i+1}$ (resp., $f_i=E_{i+1,i}$) that generate the Lie subalgebra of strictly upper (resp., lower) triangular matrices - the last exercise in Sec 1.2 of Lec 13.

Conclusion: e_i, h_i, f_i $(i=1,\dots,n-1)$ generate $\mathfrak{sl}_n(\mathbb{F})$. Now we determine the relations.

For $i,j\in\{1,2,\dots,n-1\}$, set $a_{ij}=\langle\alpha_i,h_j\rangle=\begin{cases}2, & i=j\\ -1, & i-j=\pm1\\ 0, & \text{else}\end{cases}$

Lemma: The following hold: CS

(i) "weight relations": $[h_j,e_i]=a_{ij}e_j$, $[h_j,f_i]=-a_{ij}f_j$, $\forall\, i,j$.

(ii) "$\mathfrak{sl}_2$-relation": $[e_i,f_i]=h_i$, $\forall\, i$.

(iii) e-f-relations: $[e_i,f_j]=0$, $\forall\, i\neq j$.

(iv) e-e & f-f relations: $ad(e_j)^{1-a_{ij}}e_i=ad(f_j)^{1-a_{ij}}f_i=0$, $i\neq j$.

Proof: exercise - a direct computation. Alternatively:

(i) - from the definition of roots α_i

(ii) - easy computation

(iii): $[e_i, f_j]$ has weight $\alpha_i - \alpha_j$ & this wt space is zero.

(iv) Follows from the following 3 observations (for f, e is similar)

- e, h, f span $\mathfrak{sl}_2$.
- By (iii), $\mathrm{ad}(e_j)$ kills f_i & the weight of f_i is $-a_{ij}$.
- f_i lies in a finite dimensional $\mathrm{Span}_{\mathbb{F}}(e_j, h_j, f_j)$-stable subspace, then we can use the classification of finite dimensional $\mathfrak{sl}_2$-reps to deduce the f-part of (iv) $\square$

Set $A := (a_{ij})$ and let $\mathfrak{g}(A)$ be the Lie algebra w. generators e_i, h_i, f_i and relations (i)-(iv). We get a Lie algebra epimorphism $\mathfrak{g}(A) \twoheadrightarrow \mathfrak{g}$.

Theorem: This is an isomorphism.

The proof is morally similar to that of the main theorem in Sec 1.3. of Lec 13, see Sec 4, Ch. 8 in [B]; Sec 3 in Ch. 4 of [OV], Sec 18 in [H1] (all refs are for Part 2). It's omitted.

2.2) Cartan matrices, Kac-Moody algebras & Dynkin diagrams.

Based on the theorem, we have a recipe of producing Lie algebras starting from matrices $A = (a_{ij})_{i,j \in I}$ (I is an index set) w. $a_{ij} \in \mathbb{Z}$; A should be subject to the following

(I) $a_{ii}=2 \ \forall i$ (this tells us that e_i, h_i, f_i span $\mathfrak{sl}_2$)

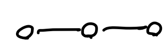
(II) $a_{ij} \leq 0$ for $i \neq j$ (otherwise (iv) doesn't really make sense)

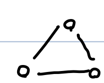
(III) $a_{ij}=0 \Leftrightarrow a_{ji}=0$ (same reason).

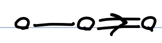
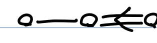
Definition: • A square matrix satisfying (I)-(III) is called a (generalized) Cartan matrix.

Given a Cartan matrix A, we can form a Lie algebra $\mathfrak{g}(A)$ w. generators e_i, h_i, f_i $(i \in I)$ and relations (i)-(iv) of Sec 1.1. This is the Kac-Moody Lie algebra $\mathfrak{g}(A)$ associated to A.

There is a way to represent a Cartan matrix as a diagram. The nodes are the elements of I. The nodes i & j are connected by $\max(-a_{ij}, -a_{ji})$ edges. If $a_{ji}=-1>a_{ij}$, we put sign $>$ in the direction $i \to j$. If $\forall i \neq j$, we have $a_{ij}=a_{ji}$ or $\max(a_{ij}, a_{ji})=-1$ (the most interesting case), then the diagram, the Dynkin diagram of A, recovers A uniquely, otherwise, there's ambiguity.

Example: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \rightsquigarrow$  (A_2), $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \rightsquigarrow$  (A_3),

$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \rightsquigarrow$  $(\tilde{A}_1)$, $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \rightsquigarrow$  $(\tilde{A}_2)$

$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \rightsquigarrow$  (B_3), $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \rightsquigarrow$  (C_3),

$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \rightsquigarrow$  (G_2). $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rightsquigarrow$  $(A_1 \times A_1)$

3) Complement: proof of the Tits deformation principle.

We start with "lifting of idempotents."

Proposition 1: Let $\mathbb{F}$ be a field, A an $\mathbb{F}[[t]]$-algebra that is a free finite rank $\mathbb{F}[[t]]$-module. Set $A_0 = A/tA$. Suppose $e_0 \in A_0$ is an idempotent, i.e $e_0^2 = e_0$. Then $\exists\, e \in A$ s.t. $e + tA = e_0$ & $e^2 = e$.

Proof: We lift "order by order": suppose $e_{k-1} \in A/t^kA$ satisfies $e_{k-1}^2 = e_{k-1}$. We claim $\exists\, e_k \in A/t^{k+1}A$ mapping to e_{k-1} & $e_k^2 = e_k$. Note that $A_0 \xrightarrow[\sim]{t^k\cdot} t^kA/t^{k+1}A$ b/c A is free over $\mathbb{F}[[t]]$. Fix some lift $\bar{e}_{k-1}$ of e_{k-1} in $A/t^{k+1}A$ so that $\bar{e}_{k-1} - \bar{e}_{k-1}^2 = t^k a$ for $a \in A_0$. We look for e_k in the form $\bar{e}_{k-1} + t^k b$. Then $(\bar{e}_{k-1} + t^k b)^2 = \bar{e}_{k-1}^2 + t^k(e_0 b + b e_0)$ should be equal to $\bar{e}_{k-1} + t^k b \Leftrightarrow a + e_0 b + b e_0 = b$. Note that $\bar{e}_{k-1} - \bar{e}_{k-1}^2 = t^k a$ implies $t^k e_0 a = t^k a e_0 \Leftrightarrow e_0 a = a e_0$. We take $b = (1-e_0) a (1-e_0) - e_0 a e_0$. It satisfies $a + e_0 b + b e_0 = b$.

There is a unique element $e \in A$ s.t. $e + t^{k+1}A = e_k$. It satisfies the required conditions. $\square$

Proposition 2: Suppose that in the previous proposition, A_0 is the direct sum of matrix algebras. Then we have an algebra isomorphism $A \to A_0 \otimes \mathbb{F}[[t]]$.

Proof: Let $A_0 = \bigoplus_{i=1}^{k} \operatorname{End}_{\mathbb{F}}(V^i)$. Pick a primitive (i.e. rk 1) idempotent $e_0^i \in \operatorname{End}(V^i)$ and lift it to $e^i \in A$. We get A-modules Ae^i

and hence an algebra homomomorphism $A \to \tilde{A} := \bigoplus_{i=1}^{k} \mathrm{End}_{\mathbb{F}[[t]]}(Ae^i)$
Note that, for a fin. gen'd $\mathbb{F}[[t]]$-module being free is equivalent to being torsion-free. Hence $Ae^i (\subset A)$ is free over $\mathbb{F}[[t]]$. Moreover, $Ae^i/tAe^i \simeq V^i$. So, it's enough to show that $A \to \tilde{A}$ is an isomorphism. Modulo t, this homomorphism gives $A_0 \to \bigoplus_{i=1}^{k} \mathrm{End}_{\mathbb{F}}(V^i)$, an isomorphism. In particular, $A \twoheadrightarrow \tilde{A}$, by the Nakayama lemma. Next $\tilde{A}$ is a free $\mathbb{F}[[t]]$-module. So, as the epimorphism of $\mathbb{F}[[t]]$-modules, $A \twoheadrightarrow \tilde{A}$ splits: $A \simeq_{\mathbb{F}[[t]]} \tilde{A} \oplus K$. Recalling that $A/tA \simeq \tilde{A}/t\tilde{A}$, we see that $K/tK = 0$ thus getting $K = 0$. $\square$

Proof of Theorem in Sec 1.2.

The rest of the proof is some algebro-geometric manipulation. Let V be a finite dimensional vector space over an algebraically closed field $\mathbb{F}$. The set of all associative bilinear products $V \times V \to V$ is a closed subvariety in $\mathrm{Hom}_{\mathbb{F}}(V \otimes V, V)$. Denote it by X. The group $GL(V)$ acts on X and the orbits are isomorphism classes of algebras.

We now produce a polynomial map $\mu: \mathbb{F} \to X$. Choose a basis in the free $\mathbb{F}[t]$-module A, say $v_1, \dots v_n$. The map μ is the multiplication table of A in this basis, i.e. $\mu(\gamma)$ is the multiplication table of $A := A/(t-\gamma)A$ for $\gamma \in \mathbb{F}$.

Let Y° denote the orbit corresponding to the isomorphism class of A_α. Let Y denote its closure in the Zariski topology. A basic fact is that Y° is Zariski open in Y.

We know $\mu(\alpha) \in Y^\circ$ and it's enough to show $\operatorname{im} \mu \subset Y$. Then $\mu^{-1}(Y^\circ)$ is Zariski open in $\mathbb{F}$ and we use that $\mathbb{F}$ is an irreducible variety to conclude that $\mu(\alpha), \mu(\beta) \in Y_\circ \Rightarrow A_\alpha \sim A_\beta$, an isomorphism of associative algebras.

Pick $f \in \mathbb{F}[\operatorname{Hom}_{\mathbb{F}}(V \otimes V, V)]$ w. $f|_Y = 0$. We need to check $\mu^*(f) = 0$. For this, we need to show that the image of $\mu^*(f)$ in $\mathbb{F}[[t-\alpha]]$ is zero (b/c $\mathbb{F}[t] \hookrightarrow \mathbb{F}[[t-\alpha]]$). This image is f evaluated at the multiplication table of $\mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}[t]} A$ in the basis $1 \otimes v_i$. Since we have an algebra isomorphism $\mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}[t]} A \simeq \mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}} A_\alpha$, we see that this multiplication table is obtained from that of A_α by applying an element of $GL_n(\mathbb{F}[[t-\alpha]])$. In other words, $\exists\, g(t) \in GL_n(\mathbb{F}[[t-\alpha]])$ s.t. the multiplication table of $\mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}[t]} A$ is $g(t)\mu(\alpha)$. Our claim is that $f(g(t)\mu(\alpha)) = 0$.

On the other hand we know that $f(g\mu(\alpha)) = 0 \ \forall g \in GL_n(\mathbb{F})$. We can view $g \mapsto f(g\mu(\alpha))$ as a polynomial in the matrix coefficients of g (and the inverse of the determinant). It vanishes. But $f(g(t)\mu(\alpha))$ is the same polynomial (but now in the matrix coefficients & $\det^{-1}$ for $g(t)$). It has to vanish. It follows that $\mu^*(f) = 0$ and completes the proof. $\square$

Rem: Here is the intuition behind the proof. We want to show $\mu(\mathbb{F}) \subset Y$.

We view $\mathbb{F}[[t-\alpha]]$ as the algebra of polynomial functions on a "tiny neighborhood" of α in $\mathbb{F}$. The isomorphism $\mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}[t]} A \xrightarrow{\sim} \mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}} A_\alpha$ can be interpreted as saying that the image of the tiny neighborhood under μ in X lies in Y.° This implies that $\mu(\mathbb{F}) \subset Y$.