# Representations of symmetric groups, part 2.

1) Centralizer subalgebra, cont'd.

2) Branching graph.

1) Recap: $B \subset A$ - fin. dim. s/simple assoc. alg's

$U \in \mathrm{Irr}(B)$, $V \in \mathrm{Irr}(A) \rightsquigarrow M_{V,U} := \mathrm{Hom}_B(U,V)$

$Z_B(A) = \{a \in A \mid ab = ba \ \forall\, b \in B\}$

**Lemma:** $Z_B(A) \xrightarrow{\sim} \bigoplus_{U,V} \mathrm{End}_{\mathbb{C}}(M_{V,U})$

**Cor:** $Z_B(A)$ is commutative $\Leftrightarrow \dim M_{V,U} \leq 1 \ \forall\, V, U$.

$\rightsquigarrow V = \bigoplus U$, uniquely.

1.1) $Z_m(n)$.

$B = \mathbb{C}S_m \subset A = \mathbb{C}S_n$ $(m<n)$, $S_m = \{\sigma \in S_n \mid \sigma(m+1) = m+1, \ldots$
$\sigma(n) = n\}$

$Z_m(n) := Z_{\mathbb{C}S_m}(\mathbb{C}S_n)$.

- Vector space basis:

**Lemma:** $H \subset G$, finite groups, $Z_{\mathbb{C}H}(\mathbb{C}G) = \{\sum_g a_g g \mid$
$a_g = a_{hgh^{-1}} \ \forall\, g \in G, h \in H\}$. So $Z_{\mathbb{C}H}(\mathbb{C}G)$ has basis indexed by $H$-conj. classes: such $c \mapsto b_c = \sum_{g \in c} g$.

Proof: $\sum_g a_g g \in Z_{\mathbb{C}H}(\mathbb{C}G) \Leftrightarrow h\sum_g a_g g = \sum_g a_g g h \Leftrightarrow$

$\sum_g a_g g = \sum_g a_g h^{-1}gh = \sum_g a_{hgh^{-1}} g \Leftrightarrow a_g = a_{hgh^{-1}}\ \forall h\in H, g\in G$ $\square$

Now $H=S_m$, $G=S_n$: $H \curvearrowright G$ – by permuting entries $1,\dots m$
So $H$-conjugacy classes $\overset{\sim}{\longleftrightarrow}$ cycle types w. marked el'ts $m+1,\dots,n$.

E.g. in $S_6$ have $S_3$-conj. classes: $* \in \{1,2,3\}$.
$(**4)(5*) \ni (1,2,4)(5,3)$ or $(2,3,4)(5,1)$
$(**5)(46) \ni (1,2,5)(4,6)$.

Example: $m=n-1$, $c=(*n)=\{(1,n),(2,n),\dots,(n-1,n)\}$.
$\underset{J_n}{\overset{\shortparallel}{b_c}} = \sum_{i=1}^{n-1}(i,n) \in Z_{n-1}(n)$ – Jucys-Murphy element

• Generators of $Z_m(n)$ (as an algebra)
$Z_m(n)$ contains the following subalgebras/elements.

(a) $Z_m(m) = $ (center of $\mathbb{C}S_m$) $= Z_m(n)\cap \mathbb{C}S_m$ – central subalgebra of $Z_m(n)$: $z\in Z_m(n)$, $x\in Z_m(m)\subset \mathbb{C}S_m$ $\Rightarrow zx=xz$.

(b) $S_{[m+1,n]} = \{\sigma\in S_n \mid \sigma(1)=1,\dots,\sigma(m)=m\}$, $\forall$ el't of $S_{[m+1,n]}$ commutes w. $\forall$ el't of $S_m \Rightarrow \mathbb{C}S_{[m+1,n]}\subset Z_m(n)$.

(c) JM el'ts $J_k = \sum_{i=1}^{k-1} (i,k) \subset Z_{k-1}(k) \subset Z_m(n)$ for $k = m+1, \dots, n$.

Rem: $J_{m+1}, \dots, J_n$ pairwise commute: $k<\ell \Rightarrow J_k \in \mathbb{C}S_{\ell-1}$, $J_\ell \in Z_{\ell-1}(\ell) \Rightarrow J_\ell J_k = J_k J_\ell$.

Thm: As algebra, $Z_m(n)$ is generated by $Z_m(m)$, $\mathbb{C}S_{[m+1,n]}$, $J_{m+1}, \dots, J_n$.

Proof - see RT1.

Cor: The following are true:

(1) $Z_{n-1}(n)$ is commutative

(2) $\forall\, U \in Irr(\mathbb{C}S_{n-1})$, $V \in Irr(\mathbb{C}S_n) \Rightarrow \dim M_{V,U} = 0$ or $1$.

(3) $\forall$ $S_{n-1}$-irrep $U \subset V$, $J_n$ acts on $U$ by scalar (depend. on $U$).

Proof: (1): by Thm, $Z_{n-1}(n)$ is generated by: $J_n$ & $Z_{n-1}(n-1)$ - central. So generators pairwise commute $\Rightarrow$ $Z_{n-1}(n)$ is comm've.

(2): $\Leftarrow$ (1) + Corollary in Recap.

(3): $J_n$ commutes w. $\mathbb{C}S_{n-1} \Rightarrow$ operator of mult'n by $J_n$ $J_{n,V}: V \to V$ is $\mathbb{C}S_{n-1}$-linear: $J_{n,V} \in Hom_{\mathbb{C}S_{n-1}}(V,V) =$

$= [V = \oplus U]$

$= Hom_{\mathbb{C}S_{n-1}}(\oplus U, \oplus U') = \bigoplus_{U,U'} Hom_{\mathbb{C}S_{n-1}}(U,U') =$

$= [\text{Schur lemma: } Hom_{S_{n-1}}(U,U') = \begin{cases} 0, & U \not\simeq U' \\ \mathbb{C}, & U \simeq U' \end{cases}] = \bigoplus_U \mathbb{C}\, id_U \Rightarrow (3)$ $\square$

Example: 1) $V = \mathrm{refl}_n = \{(x_1, .., x_n) \in \mathbb{C}^n \mid x_1 + \ldots + x_n = 0\}$

$\|$ $n > 2$; as reps of $S_{n-1}$.

$\{(x_1, \ldots x_{n-1}, 0) \mid x_1 + \ldots + x_{n-1} = 0\} \simeq \mathrm{refl}_{n-1}$

$\oplus$

$\{(x, \ldots, x, -(n-1)x\} \simeq \mathrm{triv}_{n-1}$

$J_n = \sum_{i=1}^{n-1} (i, n) : V \to V$, $(x_1, \ldots x_n) \mapsto ((n-2)x_1 + x_n, \ldots, (n-2)x_{n-1} + x_n, x_1 + \ldots + x_{n-1})$.

So on $\mathrm{refl}_{n-1}$, $J_n$ acts by $n-2$.

on $\mathrm{triv}_{n-1}$, — · — · — by $-1$.

2) $n = 4$, $V = \mathbb{C}^2$: $S_4 \twoheadrightarrow S_3$ : $(1,2), (3,4) \mapsto (1,2)$; $(2,3) \to (2,3)$

so $S_3 \hookrightarrow S_4 \twoheadrightarrow S_3$ is the id.

$V = \mathbb{C}^2$ is pulled back from $\mathrm{refl}_3 \Rightarrow$ rest'n to $S_3$ is $\mathrm{refl}_3$

And $J_4$ acts on $V$ by $0$ – exercise.

## 2) Branching graph.

$V^n \in \mathrm{Irr}(\mathbb{C}S_n)$, $V^n = \bigoplus_{V^{n-1}} \underbrace{V^{n-1}}$ ← uniquely det.'d as subspaces

$\|$

$\underbrace{\bigoplus_1 \text{irred. } \mathbb{C}S_{n-2}\text{-modules}}$

$\|$

$\bigoplus \ldots$

$\rightsquigarrow$ for $m < n$ get $V^n = \bigoplus$ irred. $\mathbb{C}S_m$-modules

**Definition**: The branching graph is a directed graph:

- vertices $\coprod_{n \geq 1} \mathrm{Irr}(\mathbb{C}S_n)$

• edges: single edge from $U$ to $V$ if $V \in \mathrm{Irr}(\mathbb{C}S_n)$, $U \in \mathrm{Irr}(\mathbb{C}S_{n-1})$ & $U$ occurs in $V$ (for some $n$).

Ex (piece w. $n \leq 4$)

$$\begin{tikzcd}[column sep=small]
 & & & & \mathrm{triv}_1 \arrow[dll] \arrow[drr] & & & & \\
 & & \mathrm{triv}_2 \arrow[dl] \arrow[drr] & & & & \mathrm{sgn}_2 = \mathrm{refl}_2 \arrow[dll] \arrow[dr] & & \\
 & \mathrm{triv}_3 \arrow[dl] \arrow[dr] & & & \mathrm{refl}_3 \arrow[dll] \arrow[d] \arrow[drr] & & & \mathrm{sgn}_3 \arrow[dl] \arrow[dr] & \\
\mathrm{triv}_4 & & \mathrm{refl}_4 & & \mathbb{C}^2 & & \mathrm{refl}_4 \otimes \mathrm{sgn}_4 & & \mathrm{sgn}_4
\end{tikzcd}$$

Terminology: • $m < n$, $V^m \in \mathrm{Irr}(\mathbb{C}S_m)$, $V^n \in \mathrm{Irr}(\mathbb{C}S_n)$

paths $\mathrm{Path}(V^m, V^n) = \{V^m \to V^{m+1} \to \ldots \to V^n\}$

$\mathrm{Path}(V^n) := \mathrm{Path}(V^1, V^n)$

$\mathrm{Path}_n = \coprod_{V^n} \mathrm{Path}(V^n)$.

• $\overline{P} \in \mathrm{Path}(V^m, V^n) \rightsquigarrow \mathbb{C}S_m$-submodule $V^m(\overline{P}) \subset V^n$, the image of $V^m$ under embeddings determined by $\overline{P}$:

$\overline{P} = (V^m \to V^{m+1} \to \ldots \to V^n) \rightsquigarrow V^m \hookrightarrow V^{m+1} \hookrightarrow V^{m+2} \hookrightarrow \ldots \hookrightarrow V^n$

$$(1)\quad V^n = \bigoplus_{V^m \in \mathrm{Irr}(S_m)} \quad \bigoplus_{\overline{P} \in \mathrm{Path}(V^m, V^n)} V^m(\overline{P}).$$

• $\varphi_{\overline{P}}$ = the embedding $V^m \hookrightarrow V^n$ according to $\overline{P}$,

$\mathrm{im}\,\varphi_{\bar P} = V^m(\bar P)$, defined uniquely up to scaling.

**Definition:** The weight of $\bar P$, $w_{\bar P} = (w_{m+1}, \dots w_n)$, where $w_i$ is the scalar by which $J_i$ acts on $V^{i-1} \subset V^i$ $(\bar P = (V^m \to V^{m+1} \to \dots \to V^n))$ see (3) of Corollary.

**Lemma:** (1) The elements $\varphi_{\bar P}$ form basis in $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$ for $\bar P$ runs over paths in $\mathrm{Path}(V^m, V^n)$.

(2) Recall: $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$ is a module over $Z_m(n) \ni J_{m+1}, \dots, J_n$. Then $J_i \varphi_{\bar P} = w_i \varphi_{\bar P}$ $(w_{\bar P} = (w_{m+1}, \dots, w_n))$.

Proof: (1): $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n) = [V^n = \bigoplus_{V'^m} \bigoplus_{\bar P \in \mathrm{Path}(V'^m, V^n)} V'^m(\bar P)]$

$= [\text{Schur lemma}] = \bigoplus_{\bar P \in \mathrm{Path}(V^m, V^n)} \mathbb{C}\varphi_{\bar P}$ (compare: proof of (3) in Cor)

(2): Use Rem 2.5 in [RT1]: $[J_i \varphi_{\bar P}](u) = J_i[\varphi_{\bar P}(u)]$ $(\forall u \in V^m)$. But $\varphi_{\bar P}(u) \in V^m(\bar P)$ and $J_i$ acts on $V^m(\bar P)$ by $w_i \Rightarrow J_i \varphi_{\bar P} = w_i \varphi_{\bar P}$. $\square$

Special case: $m=1 \rightsquigarrow V^1 = \mathbb{C}$, $\mathbb{C}S_1 = \mathbb{C} \Rightarrow$
$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V^n) \xrightarrow{\sim} V^n$; $P \in \mathrm{Path}(V^n) \rightsquigarrow v_P := \varphi_P$ viewed as el't of $V^n$.

**Cor:** The elements $v_P$ form a basis in $V^n$, $J_i v_P = w_i v_P$ $\forall$ $i = 1, \dots, n$, where $w_P = (w_1, \dots w_n)$.

**Rem:** $J_1 = 0 \Rightarrow w_1 = 0$.

See Ex 3.4 in [RT1] for $V^n = \mathrm{refl}_n$ (or exercise).

**Cor:** $m<n$, $\underline{P} \in \mathrm{Path}(V^m)$, $\overline{P} \in \mathrm{Path}(V^m, V^n)$ ⇝ concatenate $P = (\underline{P}\,\overline{P}) \in \mathrm{Path}(V^n)$. Then $v_P$ is proportional to $\varphi_{\overline{P}}(v_{\underline{P}})$ $(v_{\underline{P}} \in V^m)$

Proof: Both lie in $V^1(P)$, 1-dim'l & $v_P$, $\varphi_{\overline{P}}(v_{\underline{P}})$ are nonzero.

$\square$