Hecke algebra/category, part III.

1) Kac-Moody algebras, cont'd.
2) Weyl & Coxeter groups and their Hecke algebras.
3) Complements.

1) In lecture 19 we have introduced Cartan matrices $A=(a_{ij})_{i,j\in I}$, their Dynkin diagrams, and the Kac-Moody Lie algebra $\mathfrak{g}(A)$ (Sec. 2.2 there). Our question for now: which interesting Lie algebras arise in this way (in Sec 2.1, Lec 19, we've seen that $\mathfrak{sl}_n$ does)

Definition: Say that A is connected if its Dynkin diagram is connected $\Leftrightarrow$ the index set I cannot be partitioned as $I_1 \amalg I_2$ w. $a_{ij}=0$ for $i\in I_1, j\in I_2$.

Exercise: If $A=\mathrm{diag}(A_1,A_2)$, then $\mathfrak{g}(A)=\mathfrak{g}(A_1)\oplus\mathfrak{g}(A_2)$.

So one can restrict to the case when A is connected, which is what we assume below. We will also need the following classes of Cartan matrices.

Definition: • We say A is symmetrizable if $\exists\ d_i\in\mathbb{Z}_{>0}, i\in I$, s.t. for $D=\mathrm{diag}(d_1,\dots d_n)$, DA is symmetric.
• We say A is of finite type if DA is positive definite.

• We say A is of affine type if DA is positive semidefinite & $\dim \ker A = 1$.

Example: 1) First we give an example of a non-symmetric Cartan matrix of finite type. Consider the Euclidian space $\mathbb{R}^n$ w. tautological orthonormal basis $\varepsilon_1, \ldots \varepsilon_n$. Set $\alpha_i = h_i := \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots n-1$ & $\alpha_n = \varepsilon_n$, $h_n = 2\varepsilon_n$. Note that $h_i := \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$. Set

$$A = ((\alpha_i, h_j))_{i,j=1}^n = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & & & -2 \\ 0 & & & -1 & 2 \end{pmatrix}.$$ This is the Cartan matrix of type B_n.

For $D = \operatorname{diag}\left(\frac{2}{(\alpha_i, \alpha_i)}\right)$, the matrix DA is the Gram matrix $((h_i, h_j))$, symmetric.

2) Now an affine type example. Consider the elements $\alpha_i = h_i = \varepsilon_i - \varepsilon_{i+1}$ as above and set $\alpha_0 = h_0 = \varepsilon_n - \varepsilon_1$ so that $\sum_{i=0}^{n-1} \alpha_i = 0$. Then $A = ((\alpha_i, h_j))$ (type $\tilde{A}_n$) is of affine type (w. $\ker A = \{(x, \ldots x)\}$).

Here's why finite type Cartan matrices are important.

Theorem: $A \mapsto \mathfrak{g}(A)$ defines a bijection between:

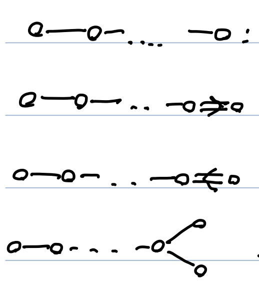
(1) Cartan matrices of finite type.

(2) Finite dimensional simple Lie algebras (over $\mathbb{C}$).

The proof of this theorem is about 2 month of varied Math... One gets from (2) to (1) in the same way as for $\mathfrak{sl}_n$: $\forall$ simple Lie algebra $\mathfrak{g}$, one has Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, bases

$\alpha_i \in \mathfrak{h}^*$ (simple roots) & $h_j \in \mathfrak{h}$ and sets $A = (\langle \alpha_i, h_j \rangle)$.

One can combinatorially classify all Cartan matrices of finite type - or the corresponding Dynkin diagrams. Here's the result. The subscript is always the number of vertices:



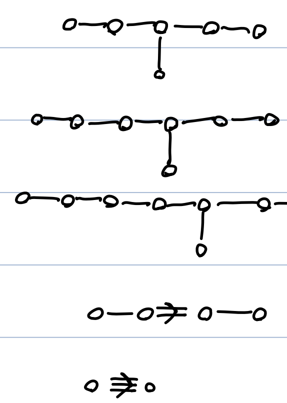
$A_n \quad (n \geq 1)$

$B_n \quad (n \geq 2)$

$C_n \quad (n \geq 3)$

$D_n \quad (n \geq 4)$

these restrictions are to avoid the repetitions.



E_6

E_7

E_8

F_4

G_2

The algebra corresponding to A_n is $\mathfrak{sl}_n$.

Optional exercise: use the complement section of Lecture 12 to verify that the matrices $(\langle \alpha_i, h_j \rangle)_{i,j=1}^n$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$ correspond to the Dynkin diagrams B_n, C_n, D_n above.

The algebras $\mathfrak{g}(A)$ for A of affine type are called affine. They appear in many parts of Math (see [Ka] e.g. in Number theory & Math Physics) and in the modular representation theory of symmetric groups, as may be explained in a bonus lecture. A concrete realization of the Lie algebra corresponding to $\tilde{A}_n$, known as $\hat{\mathfrak{sl}}_n$ is explained in the complement section.

2) Weyl & Coxeter groups and their Hecke algebras.

2.1) Weyl groups.

Let A be a Cartan matrix. Define the Cartan space $\mathfrak{h}$ w. basis $h_i, i\in I$. It maps to $\mathfrak{g}(A)$ and, in fact, the map is an embedding.

For $i\in I$ define the simple reflection $s_i\in GL(\mathfrak{h})$ by

$$s_i h_j = h_j - a_{ij}h_i$$

From $a_{ii}=2$ we deduce $s_i h_i = -h_i \Rightarrow s_i^2=1$.

Definition: The Weyl group $W(=W(A))$ is the subgroup of $GL(\mathfrak{h})$ generated by the simple reflections.

Example 0) For A of type A_n we recover the Weyl group of $\mathfrak{sl}_n$, i.e. S_n.

1) Take A of type B_n: $s_i(x)=x-(\alpha_i,x)h_i$. Then $s_i, i=1,..n-1$ act on $\mathfrak{h}=\mathbb{C}^n$ by permuting the coordinates, while s_n sends $(x_1,...x_n)$ to $(x_1,..x_{n-1},-x_n)$. It follows that $W(B_n)$ is the group of "signed permutations" & $W(B_n)\simeq S_n\ltimes\{\pm1\}^n$. Note that on $\mathfrak{h}_{\mathbb{R}}=\mathbb{R}^n$, s_i are the orthogonal reflections about codim 1 hyperplanes: $x_i=x_{i+1}$ for $i<n$ & $x_n=0$ for $i=n$.

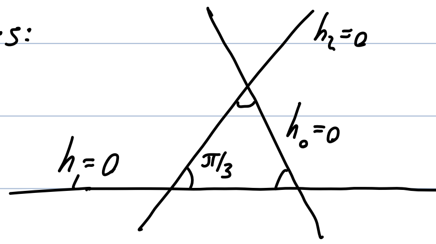
2) Take A of type $\tilde{A}_n$. Note that $\delta:=\sum_{i=0}^{n-1}h_i\in\mathfrak{h}$ is s_i-, hence W-invariant. Consider $W\curvearrowright\mathfrak{h}^*$. It preserves the affine hyperplane $\delta^{-1}(1)$. It also preserves $\mathfrak{h}^*_{\mathbb{R}}:=\mathrm{Span}_{\mathbb{R}}(h_i)^*$, hence $\delta^{-1}(1)_{\mathbb{R}}:=\delta^{-1}(1)\cap\mathfrak{h}^*_{\mathbb{R}}$. The symmetric form on $\mathfrak{h}_{\mathbb{R}}$ defined by $(h_i,h_j):=a_{ij}$ is positive semi-definite w. ker $=\mathbb{R}\delta$.

So it defines a positive definite form $\mathfrak{h}_{\mathbb{R}}/\mathbb{R}\delta$ and hence on its dual, $\delta^{-1}(0)_{\mathbb{R}}$. So $\delta^{-1}(1)_{\mathbb{R}}$ becomes a Euclidian affine space & we can talk about orthogonal reflections about <u>affine</u> hyperplanes. C6

Exercise: • s_i acts on $\delta^{-1}(0)_{\mathbb{R}}$ as a reflection about $h_i=0$, $i=0,\dots n-1$

• W is identified w. $S_n \ltimes \mathrm{Span}_{\mathbb{Z}}(\langle\alpha_i,\cdot\rangle)$, where S_n is generated by $s_1,\dots s_{n-1}$ & $\langle\alpha_i,\cdot\rangle\in\mathfrak{h}^*$ is defined by $h_j\mapsto a_{ij}$ – this is an element of $\delta^{-1}(0)_{\mathbb{R}}$ and its action of $\delta^{-1}(1)_{\mathbb{R}}$ is by translation.

You may want to consider the example of $n=3$, where the hyperplanes $h_i=0$ are as follows:



In the general case the angle between $h_i=0$ & $h_j=0$ is $\frac{\pi}{3}$ if $i-j=\pm1 \mod n$ and $\frac{\pi}{2}$, else.

2.2) Coxeter groups.

One can ask to find defining relations between the s_i's. For $i\neq j\in I$, define m_{ij} as follows

$a_{ij}a_{ji}$	0	1	2	3	$\geqslant 4$
m_{ij}	2	3	4	6	∞

Exercise: We have $(s_is_j)^{m_{ij}}=1$ if $m_{ij}\neq\infty$.

Theorem: The group $W(A)$ is generated by s_i w. relations $s_i^2=1$ & $(s_is_j)^{m_{ij}}=1$ if $m_{ij}\neq\infty$.

This is Proposition 3.13 in [Ka].

Fix a finite set I and $m_{ij}\in\mathbb{Z}_{\geqslant 2}\sqcup\{\infty\}$, $m_{ij}=m_{ji}$ $(i\neq j\in I)$. Define the group W with relations as in Theorem. These are so called Coxeter groups. The finite Coxeter groups are exactly the finite groups of isometries of Euclidian spaces that are generated by reflections (about codim 1 hyperplanes). They include

- the finite Weyl groups - that are characterized by the property that they preserve an integral lattice.
- The dihedral groups $I_2(m)=\langle s_1,s_2\rangle/(s_1^2=s_2^2=(s_1s_2)^m=1)$ that are Weyl groups exactly for $m=2,3,4,6$ (types $A_1\times A_1$, A_2, B_2, G_2)
- Two more exceptional groups, H_3, H_4.

For a detailed treatment of Coxeter groups, Weyl groups, root systems (that belongs to the discrete geometry and is a part of the structure theory of semisimple Lie algebras), see [B].

2.3) Generic Hecke algebras.

Let W be a Coxeter group and S be the set of simple reflections (=generators s_i). For $w\in W$ it makes sense to speak about its length $\ell(w)$.

Lemma: $\ell(sw)=\ell(w)\pm 1$, $\forall\, s\in S, w\in W$

Proof: Note that we have the homomorphism $\mathrm{sgn}: W\longrightarrow\{\pm 1\}$ w. $s\mapsto -1$ $(s\in S)$ - this respects the relations. Then $\mathrm{sgn}(w)=(-1)^{\ell(w)}$. Then we argue as in the proof of 3) of Corollary in Sec 1.1 of Lec 19. $\square$

We proceed to defining the generic Hecke algebra. For $s\in S$ pick an indeterminate t_s, where we declare $t_s=t_{s'}$ if s & s' are conjugate in W. For example, for types A_n $(W=S_{n+1})$ and $\tilde{A}_n$ (the latter for $n\geqslant 3$) all simple reflections are conjugate so we have $t_s=t$. For B_n $(W=S_n\ltimes\{\pm1\}^n)$ the reflections $s_1,\dots s_{n-1}$ are conjugate but not conjugate to s_n. So we have two indeterminates, $t=t_{s_i}$ $(i=1,\dots,n-1)$, $t':=t_{s_n}$.

Definition/Theorem: Set $R:=\mathbb{Z}[t_s \mid s\in S]$. Let $\mathcal{H}_R(W)$ be the free R-module w. basis $T_w\in W$. $\exists!$ associative product on $\mathcal{H}_R(W)$ s.t.

- $T_u T_w = T_{uw}$ if $\ell(uw)=\ell(u)+\ell(w)$.
- $T_s^2=(t_s-1)T_s+t_sT_1$, $\forall\, s\in S$.

The uniqueness part is easy and is left as an exercise. A proof the existence part will be explained in the complement.

2.4) Specializations of Hecke algebras.

1) Specializing $t_s=1$ for all s we recover $\mathbb{Z}W$.

2) Suppose W is a finite Weyl group ($= W(A)$ for a Cartan matrix A of finite type) and q is a prime power. Then the specialization to $t_s = q$ for all $s \in S$ gives the convolution algebra $(\mathbb{Z}[B(q)\backslash G(q)/B(q)], *)$ for a "split" finite group of Lie type $G(q)$ and its Borel subgroup $B(q)$. For example, equip $\mathbb{F}_q^{2n+1}$ w. orthogonal form $(x,y) = \sum_{i=1}^{2n+1} x_i y_{2n+2-i}$, take $G(q) = SO_{2n+1}(\mathbb{F}_q)$ and let $B(q)$ be the subgroup of all upper triangular matrices in $G(q)$. The relevant Hecke algebra is for $W(B_n) = S_n \ltimes \{\pm 1\}^n$.

2') Some "unequal parameters" (t_s's go to different numbers) specializations correspond to "non-split" finite groups of Lie type. The simplest example of such a group is the finite unitary group $GU_n(q)$ defined as follows. Let $\bar{\cdot}$ denote the nontrivial $\mathbb{F}_q$-linear automorphism of $\mathbb{F}_{q^2}$, it has order 2. Consider the sesquilinear form $(\cdot,\cdot)$ on $\mathbb{F}_{q^2}$: $(x,y) = \sum_{i=1}^{n} x_i \bar{y}_{n+1-i}$. By definition, $GU_n(q)$ is the subgroup of its isometries in $GL_n(\mathbb{F}_{q^2})$. The relevant Weyl group is of type B.

3) Let W be of affine type, e.g. $W = W(\tilde{A}_n)$. Then the specialization of $H_R(W)$ to $t_s = q$ (prime power) arises as the convolution algebra for a "p-adic group," e.g. for $W(\tilde{A}_n)$ the groups of interest are $SL_n(\mathbb{Q}_p)$ ($q=p$) or $SL_n(\mathbb{F}_q((t)))$. There may be a bonus lecture about this...

3.1) Complement: the affine Lie algebra $\hat{\mathfrak{sl}}_n$.

Here $\mathbb{F}$ is an algebraically closed field of characteristic 0. Set $\hat{\mathfrak{sl}}_n(\mathbb{F}) := \mathfrak{sl}_n \otimes \mathbb{F}[t^{\pm 1}] \oplus \mathbb{F}c$, with the unique $\mathbb{F}$-linear bracket satisfying:

$$[x\otimes t^k, y\otimes t^\ell] = [x,y]\otimes t^{k+\ell} + k\,\delta_{k+\ell,0}\, tr(xy)\, c$$
$$[x\otimes t^k, c] = 0, \quad \forall\, x,y\in \mathfrak{sl}_n,\ k,\ell\in\mathbb{Z}.$$

Exercise: Set $I = \{0,\dots,n-1\}$, and consider the elements

$$e_i = E_{i,i+1}\otimes 1,\ i=1,\dots,n-1,\quad e_0 = E_{n1}\otimes t$$
$$f_i = E_{i+1,i}\otimes 1,\ i=1,\dots,n-1,\quad f_0 = E_{1n}\otimes t^{-1}$$
$$h_i = E_{i,i} - E_{i+1,i+1},\ i=1,\dots,n-1,\quad h_0 = c - E_{11} + E_{nn}$$

Prove that these elements satisfy the relations of the Kac-Moody algebra $\mathfrak{g}(\tilde{A}_n)$

It turns out that $\mathfrak{g}(\tilde{A}_n) \twoheadrightarrow \hat{\mathfrak{sl}}_n$ is actually an isomorphism. The proof is similar to that of Thm in Sec 1.1 but is more complicated, see Section 7 in [Ka].

There is an issue w. this definition. Define the Cartan subalgebra $\hat{\mathfrak{h}} \subset \mathfrak{g}(A)$ as $\mathrm{Span}(h_0,\dots,h_{n-1})$. Then the simple roots $=$ ($\hat{\mathfrak{h}}$-eigenvalues of the generators e_i, $i\in I$), are easily seen to be linearly dependent. This complicates the structure & representation theory. To fix this, one considers a larger algebra $\tilde{\mathfrak{sl}}_n = \hat{\mathfrak{sl}}_n \oplus \mathbb{F}d$, where $\hat{\mathfrak{sl}}_n$ is embedded as a subalgebra & $[d,c]=0$, $[d, x\otimes t^k] = k\, x\otimes t^k$. If we define the Cartan subalgebra as $\tilde{\mathfrak{h}} = \mathrm{Span}_{\mathbb{F}}(h_0,\dots,h_{n-1},d)$, then the simple roots are linearly independent (but still do not span $\tilde{\mathfrak{h}}^*$).

An advantage of this ramification is that we now can view the simple roots α_i as elements of $\tilde{\mathfrak{h}}^*$: $\langle\alpha_i, h_j\rangle := a_{ij}$ & $\langle\alpha_i, d\rangle := \delta_{i0}$.

Let's explain which representations of $\hat{\mathfrak{sl}}_n$ (or $\tilde{\mathfrak{sl}}_n$ are interesting). There are two conditions one can impose.

Define the weight lattice $\Lambda \subset \tilde{\mathfrak{h}}^*$ as

$$\Lambda := \{\lambda \in \Lambda \mid \langle\lambda, h_i\rangle \in \mathbb{Z} \ \forall\, i=0,\dots,n-1,\ \langle\lambda, d\rangle \in \mathbb{Z}\}$$

Then we can talk about weight $\tilde{\mathfrak{sl}}_n$-modules: those M w. decomposition $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where $\tilde{\mathfrak{h}}$ acts on M_λ w. λ, and $\dim M_\lambda < \infty \ \forall \lambda$.

By a highest weight module we mean a weight module M whose weights are "bounded from above": $\exists\ \lambda_1, \dots \lambda_k$ s.t. $M_\lambda \neq 0 \Rightarrow \lambda \in \bigcup_{i=1}^{k} \lambda_i - \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\alpha_0, \dots \alpha_{n-1})$.

There are no finite dimensional weight $\tilde{\mathfrak{sl}}_n$-modules. Instead, one can consider integrable representations: those M s.t. $\forall\ m \in M\ \exists\ N (=N(m)) \in \mathbb{Z}_{\geqslant 0} \Rightarrow e_j^{\ell} m = f_j^{\ell} m = 0$, $\forall\ \ell \geqslant N$, $j = 0, \dots, n-1$. An integrable weight module may fail to be highest weight, an example is provided by the adjoint representation. But the integrable highest weight modules enjoy many similarities with their finite dimensional counterparts (for $\mathfrak{sl}_n$): they are completely reducible, the irreducibles are classified by dominant weights: $\lambda \in \Lambda$ s.t. $\langle\lambda, h_i\rangle \in \mathbb{Z}_{\geqslant 0}\ \forall\ i=0,\dots,n-1$, the characters are computed by the Weyl-Kac character formula. Details can be found in Sections 3, 9-12, [Ka].

3.2) Complete: sketch of proof of Theorem from Section 2.3.

Let M be a free R-module w. basis $b_w, w\in W$. For $s\in S$, define the operators $T_{s,L}, T_{s,R}$ on M by

$$T_{s,L}b_w = \begin{cases} b_{sw}, \text{ if } \ell(sw)=\ell(w)+1 \\ (t_s-1)b_w + t_s b_{sw}, \text{ else}\end{cases} \qquad T_{s,R}b_w = \begin{cases} b_{ws}, \text{ if } \ell(ws)=\ell(w)+1 \\ (t_s-1)b_w + t_s b_{ws}, \text{ else}\end{cases}$$

The main technical step is to check that the operators $T_{s,L}T_{t,R}=T_{t,R}T_{s,L}$ $\forall s,t\in S$. This requires the case by case argument based on the relation between $\ell(w), \ell(sw), \ell(wt), \ell(swt)$, which is performed using the following claim:

Exchange lemma: For $w\in W$ any two "reduced expressions" for w, i.e. expressions of the form $w=s_{i_1}\dots s_{i_\ell}$ w. $\ell=\ell(w)$ are obtained from one another by a sequence of "braid moves": $\underbrace{ts\dots}_{m_{st}} \leftrightarrow \underbrace{st\dots}_{m_{st}}$ (e.g. for $m_{st}=3$, we get $sts\longleftrightarrow tst$).

Once $T_{s,L}T_{t,R}=T_{t,R}T_{s,L}$ is known we construct the product as follows. For $w\in W$ w. reduced expression $w=s_{i_1}\dots s_{i_\ell}$ define $T_{w,L}: M\to M$ as $T_{s_{i_1},L}\dots T_{s_{i_\ell},L}$. It's independent of the choice of a reduced expression of w: $T_{s_{i_1},L}\dots T_{s_{i_\ell},L}b_u = T_{s_{j_k},R}\dots T_{s_{j_1},R}b_w$ $\forall$ reduced expression $u=s_{j_1}\dots s_{j_k}$. We then define the product $b_w b_u := T_{w,L}b_u$. It's associative: $(b_w b_u)b_v$ $= T_{w,L}T_{v,R}b_v = T_{u,R}T_{w,L}b_u = b_w(b_u b_v)$, where $T_{v,R}$ is defined similarly to $T_{w,L}$. $\square$

Remark: Let's show by example why we require that the conjugate reflections give the same indeterminate. Consider $W=S_3$ and the element $T_{s_1}T_{s_2}T_{s_1}T_{s_2}$. We compute it in two ways:

$$T_{s_1}T_{s_2}T_{s_1}T_{s_2} = T_{s_1}T_{s_1}T_{s_2}T_{s_1} = \big((t_{s_1}-1)T_{s}+t_{s_1}\big)T_{s_2}T_{s_1}$$

$$= T_{s_2}T_{s_1}T_{s_2}T_{s_2} = T_{s_2}T_{s_1}\big((t_{s_2}-1)T_{s_2}+t_{s_2}\big)$$

The resulting expressions are equal $\Leftrightarrow t_{s_1}=t_{s_2}$.