Hecke algebra/category, part IV.

1) Kazhdan-Lusztig basis

2) Complements.

1) For an indeterminate t we have defined (Lecture 19) the generic Hecke algebra $\mathcal{H}^{\mathbb{Z}}(W)$ (for $W=S_n$) over $\mathbb{Z}[t]$. In this lecture, we'll need a slight modification. Consider the homomorphism $\mathbb{Z}[t]\to\mathbb{Z}[v^{\pm 1}], t\mapsto v^{-2}$, and set $\mathcal{H}_v(W):=\mathbb{Z}[v^{\pm 1}]\otimes_{\mathbb{Z}[t]}\mathcal{H}^{\mathbb{Z}}(W)$. For $w\in W$, define an element $H_w:=v^{-\ell(w)}\otimes T_w\in\mathcal{H}_v(W)$. These elements form a basis of $\mathcal{H}_v(W)$ called the standard basis. Note that the product on $\mathcal{H}_v(W)$ is uniquely recovered from

(1) $H_u H_w = H_{uw}$ if $\ell(uw)=\ell(u)+\ell(w)$ $(\Rightarrow H_w = H_{s_{i_1}}\dots H_{s_{i_\ell}}$ if $w=s_{i_1}\dots s_{i_\ell}, \ell:=\ell(w))$.

(2) $H_s^2=(v^{-1}-v)H_s+1 \Leftrightarrow (H_s+v)(H_s-v^{-1})=0 \Leftrightarrow T_s^2=(t-1)T_s+t$

(1) & (2) imply

$$(3)\quad H_s H_w=\begin{cases} H_{sw} & \text{if } \ell(sw)=\ell(w)+1\\ (v^{-1}-v)H_w+H_{sw}, & \text{else}\end{cases}$$

Our goal in this lecture is to produce a different basis of $\mathcal{H}_v(W)$, the Kazhdan-Lusztig basis.

1.1) Bar involution. Our first ingredient is a certain ring automorphism $\bar{\cdot}$ of $\mathcal{H}_v(W)$. Note that each H_s is invertible in $\mathcal{H}_v(W)$ $((2)\Rightarrow H_s^{-1}=H_s+v-v^{-1})$ and hence each H_w is invertible thx to (1).

Proposition/definition: The map $x \mapsto \bar{x}$ given on $\mathbb{Z}$-basis $v^k H_w$ by $\overline{v^k H_w} = v^{-k} H_{w^{-1}}^{-1}$ is a ring automorphism called the bar involution.

Proof: We need to check that relations (1) & (2) are preserved by $\bar{\cdot}$:

(1): $\overline{H_u H_w} = \overline{H_{uw}} = H_{(uw)^{-1}}^{-1} = H_{w^{-1}u^{-1}}^{-1} = [\ell((uw)^{-1}) = \ell(uw) = \ell(u) + \ell(w) = \ell(u^{-1}) + \ell(w^{-1})]$

$[(1)] = (H_{w^{-1}} H_{u^{-1}})^{-1} = H_{u^{-1}}^{-1} H_{w^{-1}}^{-1} = \overline{H_u}\,\overline{H_w}$. ✓

(2): $(\overline{H_s} + \bar{v})(\overline{H_s} - \overline{v^{-1}}) = (H_s^{-1} + v^{-1})(H_s^{-1} - v) = H_s^{-2}(1 + v^{-1}H_s)(1 - H_s v)$

$= -H_s^{-2}(H_s + v)(H_s - v^{-1}) = 0$ ✓

□

Remark: 1) This is indeed an involution – exercise.

2) Later on we'll discuss how $\bar{\cdot}$ enters the picture – and what it has to do w. functor $\mathbb{D}$ from 3) in HW3.

1.2) Kazhdan-Lusztig basis

Theorem (essentially Kazhdan & Lusztig '1979) $\exists!$ $\mathbb{Z}[v^{\pm 1}]$-basis C_w $(w \in W)$ of $\mathcal{H}_v(W)$ (Kazhdan-Lusztig basis) s.t.

(i) $C_w = \overline{C_w}$ $\forall\, w \in W$.

(ii) $C_w \in H_w + v\,\mathrm{Span}_{\mathbb{Z}[v]}(H_u \mid u \in W)$.

The following establishes the uniqueness part.

Lemma: Let C_u $(u \in W)$ be a KL basis. Pick $w \in W$ and let C_w' be an element satisfying (i) & (ii). Then $C_w' = C_w$.

Proof: Note that (ii) $\Leftrightarrow H_u \in C_u + v\,\mathrm{Span}_{\mathbb{Z}[v]}(C_x \mid x \in W) \Rightarrow$

$C_w' = \sum_{u \in W} F_{wu}(v) C_u$ w. $F_{wu} \in \delta_{uw} + v\mathbb{Z}[v]$. Then $\overline{C_w'} = \sum F_{wu}(\bar{v}) \overline{C_w} =$

$[(i)] = \sum F_{wu}(v^{-1}) C_w \Rightarrow F_{wu}(v) = F_{wu}(v^{-1})$, contradiction. $\square$

Example: 1 & H_s+v satisfy (i) & (ii):

$$\overline{H_s+v} = [s=s^{-1}] = H_s^{-1}+v^{-1} = [H_s^{-1} = H_s + v - v^{-1}] = H_s+v.$$

So, we must have $C_1 = 1$, $C_s = H_s+v$.

1.3) Existence

We will prove the existence of a basis with stronger properties

Definition: Define the Bruhat order $\preceq$ on W by $u \preceq w$ if $\exists$ transpositions $t_1, \dots, t_k \in W$ s.t

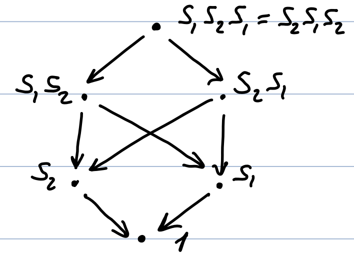
$$\ell(t_i \dots t_k w) < \ell(t_{i+1} \dots t_k w) \; \forall i = 1, \dots k$$

and $u = t_1 \dots t_k w$. Note that this is indeed a partial order.

Exercise: 1) For $t = (i,j)$ w. $i<j$, $\ell(tw) < \ell(w) \Leftrightarrow w^{-1}(i) > w^{-1}(j)$.

2) 1 is the unique min. element, and w_0 is the unique max. element.

Example: The Bruhat order on S_3 is described by the following directed graph, the Bruhat graph, ($u \preceq w$ if $\exists$ path $w \to u$)



Proof of the existence part: We'll construct C_w satisfying (i) & (ii'): $C_w = T_w + \sum_{u \prec w} v p_{wu}(v) T_u$ w. $p_{wu}(v) \in \mathbb{Z}[v]$.

The construction is recursive: for $w \in W$ suppose we've constructed C_u satisfying (i) & (ii') $\forall u \prec w$. Set $\Lambda(\prec w) = \mathrm{Span}_{\mathbb{Z}[v]}(H_u \mid u \prec w)$ and define $\Lambda(\preceq w)$ analogously. (ii') becomes $C_w \in T_w + v\Lambda(\prec w)$.

Let $w = s_{i_1} \dots s_{i_\ell}$ w $\ell = \ell(w)$. Then for $s = s_{i_1}$ have $sw \prec w$. Consider $C_s C_{sw}$. Since $\bar{\cdot}$ is an algebra homomorphism, we get that $C_s C_{sw}$ satisfies (i). Let's see if it satisfies (ii').

$$C_s C_{sw} = (H_s + v)(H_{sw} + \sum_{u \prec sw} v\, p_{sw,u}(v) H_u) = [H_s H_{sw} = H_w] =$$
$$= H_w + \underbrace{vH_{sw} + v^2 \sum_{u \prec sw} p_{sw,u}(v) H_u}_{\in v\Lambda(\prec w)} + \underbrace{\sum_{u \prec sw} p_{sw,u}(v)\, v H_s H_u}_{\Sigma_1 + \Sigma_2}$$

We split the last sum into 2 parts: w. $\ell(su) > \ell(u)$ to be denoted by Σ_1 & w. $\ell(su) < \ell(u)$: Σ_2. The reason is (3) before Sec 1.1.

• $\ell(su) > \ell(u) \Rightarrow H_s H_u = H_{su}$. Note that $u \prec sw \prec w \Rightarrow su \prec w$. Namely, let transpositions $t_1 \dots t_k$ be s.t. $u = t_1 \dots t_k sw$ & $\ell(t_i \dots t_k sw) < \ell(t_{i+1} \dots sw)$. If $\exists$ i s.t. $t_i = s$ pick i to be maximal possible and replace w with $t_{i+1} \dots t_k w$. Otherwise, notice that $su = st_1s^{-1} st_2 s^{-1} \dots st_k s^{-1} w$ & $\ell(st_i s^{-1} \dots st_k s^{-1} w) < \ell(st_{i+1} s^{-1} \dots st_k s^{-1} w)$.

It follows that $\Sigma_1 \in v\Lambda(\prec w)$.

• $\ell(su)<\ell(u) \Rightarrow vH_sH_u=(1-v^2)H_u+vH_{su}$. So Σ_2 becomes

$$\sum_{su\prec u\prec sw} p_{sw,u}(v)[(1-v^2)H_u+vH_{su}] \in \sum_{su\prec u\prec sw} p_{sw,u}(0)H_u+v\Lambda(\prec sw)=$$

$$[C_u-H_u\in v\Lambda(\prec u);\ \Lambda(\prec u),\Lambda(\prec sw)\subset\Lambda(\prec w)]\subset \sum_{su\prec u\prec sw} p_{sw,u}(0)C_u+v\Lambda(\prec w).$$

Conclusion $C_sC_{sw}\in H_w+\sum_{su\prec u\prec sw} p_{sw,u}(0)C_u+v\Lambda(\prec w)$ so

$C_w:=C_sC_{sw}-\sum p_{sw,u}(0)C_u$ satisfies (i) and (ii') □

Example: Let's compute the elements C_w for $W=S_3$. We already know (Example in Sec 1.2) $C_1=1$, $C_{s_1}=H_{s_1}+v$, $C_{s_2}=H_{s_2}+v$.

$$C_{s_1}C_{s_2}=(H_{s_1}+v)(H_{s_2}+v)=H_{s_1s_2}+v(H_{s_1}+H_{s_2})+v^2=C_{s_1s_2},$$

$$H_{s_2s_1}+v(H_{s_2}+H_{s_1})+v^2=C_{s_2s_1}$$

$$C_{s_2}C_{s_1s_2}=(H_{s_2}+v)(H_{s_1s_2}+v(H_{s_1}+H_{s_2})+v^2)=H_{s_2s_1s_2}+vH_{s_2s_1}+\overbrace{vH_{s_2}^2}^{\text{needs treatment}}+\cancel{v^2H_{s_2}}$$

$$+vH_{s_1s_2}+v^2(H_{s_1}+H_{s_2})+v^3=[vH_{s_2}^2=(1\cancel{-v^2})H_{s_2}+v]=$$

$$=\underbrace{(H_{s_2s_1s_2}+v(H_{s_1s_2}+H_{s_2s_1})+v^2(H_{s_1}+H_{s_2})+v^3)}_{=\ C_{s_2s_1s_2}}+(H_{s_2}+v)$$

1.4) Kazhdan-Lusztig conjecture.

Definition: For $u,w\in W$, let the **Kazhdan-Lusztig polynomial** $c_{wu}(v)$ be defined by $C_w=\sum_{u\in W} c_{wu}(v)H_u$ (so that $c_{ww}=1$, $c_{wu}\neq 0\Rightarrow u\preceq w$ and for $u\prec w$ we have $c_{wu}(v)=vp_{wu}(v)$).

Let $\lambda\in\Lambda_+=\{\sum_{i=1}^n \lambda_i\varepsilon_i \mid \lambda_i-\lambda_{i+1}\in\mathbb{Z}_{\geqslant 0}\}$, $\rho=\sum_{i=1}^n \frac{n+1-2i}{2}\varepsilon_i$. Recall the element $w_0\in S_n$ $(w_0(i)=n+1-i)$. We have $w_0\rho=-\rho$ so $w_0\cdot\lambda=w_0(\lambda+\rho)-\rho$ $=w_0\lambda-2\rho=:\lambda^-$.

Recall that to $\mu\in\Lambda$ we can assign the following representations of $\mathfrak{sl}_n$: the Verma module $\Delta(\mu)$ & its irreducible quotient $L(\mu)$. For $\mu=w\cdot\lambda$, the only irreducibles that can occur in $\ker[\Delta(w\cdot\lambda)\twoheadrightarrow L(w\cdot\lambda)]$ are $L(u\cdot\lambda)$ w. $u\cdot\lambda<w\cdot\lambda$, Sec 1.1 in Lec 16. If we know their multiplities, we can express the (unknown) ch $L(w\cdot\lambda)$ via (known) ch $\Delta(u\cdot\lambda)$ w. $u\cdot\lambda\leqslant w\cdot\lambda$.

Thm (Kazhdan-Lusztig conjecture (1979) proved by Beilinson-Bernstein & Brylinski-Kashiwara (1981), reproved a number of times afterwards).

- The multiplicity of $L(u\cdot\lambda)$ in $\Delta(w\cdot\lambda)$ is $c_{u,w}(1)\Rightarrow$

$$\operatorname{ch}(\Delta(w\cdot\lambda))=\sum_{u\succeq w} c_{u,w}(1)\operatorname{ch}(L(u\cdot\lambda)).$$

- $$\operatorname{ch} L(w\cdot\lambda^-)=\sum_{u\preceq w}(-1)^{\ell(w)-\ell(u)}c_{w,u}(1)\operatorname{ch}(\Delta(u\cdot\lambda^-)).$$

Note that the upper triangularity in the theorem is different from what we had before, it's stronger, as $u\prec w\Rightarrow u\cdot\lambda>w\cdot\lambda$ (exercise).

2) Complements

Kazhdan-Lusztig bases/polynomials are remarkable objects that were extensively studied since they were discovered. Yet, much is still unknown. Here's a brief account of some developments.

2.1) Properties of KL polynomials.

• Positivity: Theorem in Sec 1.4, in particular, means that $c_{u,w}(1)\geqslant 0$ $\forall u,w\in W$. More is true: $c_{u,w}\in \mathbb{Z}_{\geqslant 0}[v]$. This is completely not obvious from the construction in Sec 1.3 - or any other combinatorial construction. The claim that $c_{u,w}\in \mathbb{Z}_{\geqslant 0}[v]$ was proved by Kazhdan and Lusztig in 1980: they checked that the coefficients of $c_{u,w}$ are the dimensions of stalks of $IC(\overline{BuB}/B,\mathbb{Q})$ on BwB. A connection to the IC's (Intersection complexes) is an important ingredient of the classical proofs of the theorem.

No enumerative meanings of the coefficients of $c_{u,w}$ (or of $c_{u,w}(1)$) is known (in general) - and none is expected to exist. Still KL combinatorics has deep connections to the classical enumerative combinatorics (for example, via the theory of "cells").

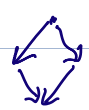
• Other restrictions: one has $c_{w,u}(v)=v^{\overbrace{\ell(w)-\ell(u)}^{\in\, \mathbb{Z}_{\geqslant 0} \text{ if } w\succ u}}P_{w,u}(v^2)$ for $P_{w,u}$, a polynomial w. integral coefficients. One can trace this from the definition. Another restriction is that the $P_{w,u}(0)=1$. And that's it: any polynomial w. non-negative integer coefficients & constant term 1 arises as $P_{w,u}$ for some $w,u\in S_n$ for some n, P. Polo, "Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups", Representation theory, 1999.

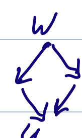
• Kazhdan-Lusztig inversion formula: Theorem in Sec. 1.4 implies $\sum_{y\in W}(-1)^{\ell(w)-\ell(y)}c_{u,y}(1)c_{yw_0,ww_0}(1)=\delta_{u,w}$ $\forall u,w\in W$. In fact, KL'79, we have $\sum_{y\in W}(-1)^{\ell(w)-\ell(y)} c_{u,y}\, c_{yw_0, ww_0} = \delta_{u,w}$.

This is a combinatorial shadow of a deep representation theoretic fact: the principal block of category $\mathcal{O}$ is Koszul self-dual. We'll mention some more on this later.

• Combinatorial invariance conjecture

A fundamental issue w. computing $c_{w,u}$ is that to compute them one needs to start with $w=1$ and do induction on the Bruhat order. At the same time, there's a lot of evidence suggesting that $c_{w,u}$ depends not on w,u themselves but on the interval between w and u in the Bruhat graph (the full subgraph, whose vertices are all vertices on a path from w to u). For example, the interval between $s_1s_2s_1$ and s_2 in the Bruhat graph looks like



and each time the interval between w and u is  we should have $c_{w,u} = v^2$. The general conjecture is known as the "combinatorial invariance conjecture", see arXiv: 2111.15161 for recent developments and more details.

2.2) Generalizations

Recall that in Section 2.2 of Lec 20 we have defined a Coxeter group W w. generators $s_i, i\in I$, and relations $s_i^2=1$

$(s_i s_j)^{m_{ij}} = 1$ for a collection of elements $m_{ij} \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$ (the relation is skipped if $m_{ij} = \infty$)

If we fix a collection of indeterminates t_i s.t. $t_i = t_j$ if s_i, s_j are conjugate (in terms of the m_{ij}'s this boils down to the condition that $t_i = t_j$ as long as m_{ij} is odd (and $<\infty$)), then we can define the generic Hecke algebra $\mathcal{H}_{(t_i)}(W)$ (Section 2.3 of Lec 20). We can consider its equal parameter specialization $\mathcal{H}_v(W) = \mathbb{Z}[v^{\pm 1}] \otimes_{\mathbb{Z}[t_i]} \mathcal{H}_{(t_i)}(W)$ w. $t_i \mapsto v^{-2}$.

We can still consider the bar-involution $\bar{\cdot}$ and define the Kazhdan-Lusztig basis $C_w, w \in W$, as in the S_n-case. The resulting basis and the corresponding KL polynomials share many similarities to the S_n-case, as the same geometric picture holds (where one replaces the usual flag variety G/B for $G = SL_n$ with the flag variety for the corresponding Kac-Moody group). For example, we still have $c_{w,u} \in \mathbb{Z}_{\geq 0}[v]$.

For the general Coxeter groups, the flag varieties aren't there. Relatively recently Elias and Williamson, "Hodge theory of Soergel bimodules", Ann. Math. (2) 180 (2014), 1089-1136, proved that $c_{w,u} \in \mathbb{Z}_{\geq 0}[v]$ for all Coxeter groups. The proof is algebraic but heavily uses geometric insights (it emulates the Hodge theory from Algebraic geometry).

More generally, we can specialize t_i's to different powers of v (when we have more than one conjugacy class, $W(B_2)$ a.k.a. the order 8 dihedral group, is the simplest example). Here we can specialize t_i's to different powers of v (usually, both negative & even). But if the powers are not the same, the positivity may fail – already in type B_2.