1) For an indeterminate $t$ we have defined (Lecture 19) the generic Hecke algebra $H^\mathbb{Z}(W)$ (for $W = S_n$) over $\mathbb{Z}[t]$. In this lecture, we'll need a slight modification. Consider the homomorphism $\mathbb{Z}[t] \to \mathbb{Z}[v^\pm], t \mapsto v^2$ and set $H_v(W) = \mathbb{Z}[v^\pm] \otimes H^\mathbb{Z}(W)$. For $w \in W$, define an element $H_w = v^{\ell(w)} \otimes T_w \in H_v(W)$. These elements form a basis of $H_v(W)$ called the standard basis. Note that the product on $H_v(W)$ is uniquely recovered from

1. $H_u H_w = H_{uw}$ if $\ell(uw) = \ell(u) + \ell(w)$ ($\Rightarrow H_w = H_{S_1} \cdots H_{S_n}$ if $w = s_{i_1} \cdots s_{i_n}, \ell = \ell(w)$).
2. $H_5^2 = (v^{-1} - v)H_5 + 1 \Leftrightarrow (H_5 + v)(H_5 - v) = 0 \Leftrightarrow T_5^2 = (t-1)T_5 + t$

(1) & (2) imply

3. $H_5 H_w = \begin{cases} H_{5w} & \text{if } \ell(sw) = \ell(w) + 1 \\ (v^{-1} - v)H_w + H_{5w}, & \text{else} \end{cases}$

Our goal in this lecture is to produce a different basis of $H_v(W)$, the Kazhdan-Lusztig basis.

1.1) Bar involution. Our first ingredient is a certain ring automorphism $\tilde{\cdot}$ of $H_v(W)$. Note that each $H_5$ is invertible in $H_v(W)$ ($\Rightarrow H_5^{-1} = H_5 + v - v^{-1}$) and hence each $H_w$ is invertible thrx to (1).
Proposition/definition: The map $x \mapsto \overline{x}$ given on $\mathbb{Z}$-basis $\mathbf{w}^*$ by $\mathbf{w}^* \mathbf{w} = \mathbf{w}^* \mathbf{w}^{-1}$ is a ring automorphism called the bar involution.

Proof: We need to check that relations (1) & (2) are preserved by $\overline{\cdot}$:

1. $\overline{H_u H_v} = \overline{H_u H_v}^{-1} = H_u^{-1} H_v^{-1} = [c(u w)^{-1} = c(u w) = c(u) + c(w) = c(u^{-1}) + c(w^{-1})]$

2. $(H_u + \overline{\mathbf{w}})(H_u - \overline{\mathbf{w}}) = (H_u^{-1} + \mathbf{w}^{-1})(H_u^{-1} - \mathbf{w}) = H_u^{-2}(1 + \mathbf{w}^{-1} H_u)(1 - H_u \mathbf{w})$

Remark: 1) This is indeed an involution - exercise.

2) Later on we'll discuss how $\overline{\cdot}$ enters the picture - and what it has to do with functor $\mathbf{D}$ from 3) in HW3.

1.2) Kazhdan-Lusztig basis

Theorem (essentially Kazhdan & Lusztig '79) $\exists! \mathbb{Z}[v^\pm]$-basis $\mathbf{w}$

(w \in W) of $H_v$ (Kazhdan-Lusztig basis) s.t.

(i) $\mathbf{w} = \overline{\mathbf{w}} \neq w \in W$.

(ii) $\mathbf{w} \in H_v + v \text{ Span}_{\mathbb{Z}[v]}(H_u | u \in W)$.

The following establishes the uniqueness part.

Lemma: Let $\mathbf{w} (w \in W)$ be a KL basis. Pick $w \in W$ and let $\mathbf{w}'$ be an element satisfying (i) & (ii). Then $\mathbf{w}' = \mathbf{w}$.

Proof: Note that (ii) $\Leftrightarrow H_v \in C_w + v \text{ Span}_{\mathbb{Z}[v]}(C_x | x \in W) \Rightarrow$

$\mathbf{w}' = \sum_{u \in W} F_{w u}(v) C_u \quad w \in S_{w u} + v \mathbb{Z}[v]$. Then $\mathbf{w}' = \sum_{u \in W} F_{w u}(v) \mathbf{w}$.
\[(ii) = \sum F_{wu}(v^{-1})C_w \Rightarrow F_{wu}(v) = F_{wu}(v^{-1}), \text{ contradiction.} \square\]

**Example:** \(1 \& H_s^t + v \) satisfy \( (i) \& (ii) : \)

\[
H_s^t + v = [s = s^{-1}] = H_s^{-1} + v^{-1} = [H_s^{-1} = H_s + v - v^{-1}] = H_s^t + v
\]

So, we must have \( C_1 = 1, C_3 = H_s^t + v. \)

**13) Existence**

We will prove the existence of a basis with stronger properties.

**Definition:** Define the Bruhat order \( \leq \) on \( W \) by \( u \leq w \) if \( \exists \)

transpositions \( t_1, ..., t_k \in W \) s.t

\[
el(t_i ... t_k w) < el(t_i, ... t_k w) \forall i = 1, ..., k
\]

and \( u = t_i ... t_k w. \) Note that this is indeed a partial order.

**Exercise:**

1) For \( t = (i, j) \) w. \( i < j, \) \( el(tw) < el(w) \iff w'(i) > w'(j). \)

2) \( 1 \) is the unique min element, and \( w_0 \) is the unique max element.

**Example:** The Bruhat order on \( S_3 \) is described by the following directed graph, the Bruhat graph, (\( u \leq w \) if \( \exists \) path \( w \rightarrow u \))

\[
\begin{align*}
S_1 & \rightarrow S_1S_2 \rightarrow S_2S_1 \\
S_2 & \rightarrow S_1S_2S_1 \rightarrow S_2S_1S_2 \\
S_1 & \rightarrow S_1S_2S_1S_2 \rightarrow S_2S_1S_2S_1 \\
S_2 & \rightarrow S_1S_2S_1S_2S_1 \rightarrow S_2S_1S_2S_1S_2 \\
S_1S_2S_1S_2S_1 & \rightarrow S_2S_1S_2S_1S_2S_1
\end{align*}
\]
Proof of the existence part: We'll construct $C_w$ satisfying (i) & (ii'): $C_w = T_w + \sum_{u \leq w} v p_{wu}(v) T_u$ $w p_{wu}(v) \in \mathbb{Z}[v]$.

The construction is recursive: for $w \in W$ suppose we've constructed $C_u$ satisfying (i) & (ii') $\forall u < w$. Set $\Lambda(\leq w) = \text{Span}_{\mathbb{Z}[v]}(H_u | u < w)$ and define $\Lambda(\leq w)$ analogously. (ii') becomes $C_w \in T_w + v \Lambda(\leq w)$.

Let $w = s_i \ldots s_t w \in C(w)$. Then for $s = s_i$ have $sw < w$. Consider $C_s C_{sw}$. Since $\circ$ is an algebra homomorphism, we get that $C_s C_{sw}$ satisfies (i). Let's see if it satisfies (ii').

$$C_s C_{sw} = (H_s + v)(H_{sw} + \sum_{u \leq sw} v p_{swu}(v) H_u) = [H_s H_{sw} = H_w] =$$

$$= H_w + v H_{sw} + v^2 \sum_{u \leq sw} p_{swu}(v) H_u + \sum_{u < sw} p_{swu}(v) v H_s H_u$$

$$\in \circ \Lambda(\leq w) \Sigma_1 + \Sigma_2$$

We split the last sum into 2 parts: $w \circ (su) \geq \circ (u)$ to be denoted by $\Sigma_1$ & $w \circ (su) < \circ (u)$: $\Sigma_2$. The reason is (3) before Sec. 1.1.

- $\circ (su) > \circ (u) \Rightarrow H_s H_u = H_{su}$. Note that $u < sw < w \Rightarrow su < w$. Namely, let transpositions $t_1 \ldots t_k$ be s.t. $u = t_1 \ldots t_k sw$ & $l(t_1 \ldots t_k sw) < l(t_1 \ldots sw)$. If $\exists i$ s.t. $t_i = s$ pick $i$ to be maximal possible and replace $w$ with $t_{im} \ldots t_k w$. Otherwise, notice that $su = st_is' st_{i+1} \ldots st_{k-1} s' w$ & $l(st_is' \ldots st_{k-1} s' w) < l(st_is' \ldots st_{k-1} s' w)$.

It follows that $\Sigma_1 \in \circ \Lambda(\leq w)$.
\* \ell(su) < \ell(u) \Rightarrow vH_sH_u = (1-v^2)H_u + vH_{su}. \text{So } \Sigma_v \text{ becomes}
\[
\sum_{su < u \times su} p_{swu}(v)[(1-v^2)H_u + vH_{su}] \leq \sum_{su < u \times su} p_{swu}(v)H_u + v\Lambda(<sw) = \\
[C_u - H_u \leq v\Lambda(<u), \Lambda(<u), \Lambda(<sw) \subseteq \Lambda(<w)] \leq \sum_{su < u \times sw} p_{swu}(v)C_u + v\Lambda(<w).
\]

**Conclusion** 
\[C_sC_{sw} \leq H_w + \sum_{su < u \times sw} p_{swu}(v)C_u + v\Lambda(<w) \text{ so}
\]
\[C_w := C_sC_{sw} - \sum_{su < u \times sw} p_{swu}(v)C_u \text{ satisfies (i) and (ii') } \square\]

**Example:** Let's compute the elements \(C_w\) for \(W = S_3\). We already know (Example in Sec 1.2) \(C_1 = 1, C_{s_1} = H_{s_1} + v, C_{s_2} = H_{s_2} + v^2\).
\[
C_{s_1}C_{s_2} = (H_{s_1} + v)(H_{s_2} + v) = H_{s_1s_2} + v(H_{s_1}H_{s_2}) + v^2 = C_{s_1s_2},
\]
\[
H_{s_1s_2} + v(H_{s_1}H_{s_2}) + v^2 = C_{s_1s_2}, \text{ needs treatment}
\]
\[
C_{s_2}C_{s_1s_2} = (H_{s_2} + v)(H_{s_1s_2} + v(H_{s_1}H_{s_2}) + v^2) = H_{s_1s_2s_1} + vH_{s_1s_2} + vH_{s_2} + v^2H_{s_2} + v^3H_{s_2} + v^3H_{s_2} + v^3 = [vH_{s_2}^2 = (1-v^2)H_{s_2} + v]
\]
\[= (H_{s_2s_1s_2} + v(H_{s_2s_1s_2} + v(H_{s_1s_2s_1} + v(H_{s_2s_1s_2} + v^3))) + (H_{s_2} + v)
\]
\[C_{s_2s_1s_2}
\]

**1.9) Kazhdan-Lusztig conjecture.**

**Definition:** For \(u, w \in W\), let the Kazhdan-Lusztig polynomial \(C_{wu}(v)\) be defined by \(C_w = \sum_{u \in W} C_{wu}(v)H_u\) (so that \(C_{wu} = 1, C_{wu} \not= 0 \Rightarrow u \preceq w\) and for \(u \preceq w\) we have \(C_{wu}(v) = v\delta_{wu}(v)\)).
Let $\lambda \in \Lambda_+ = \{ \sum_{i=1}^n \lambda_i \epsilon_i | \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda = \sum_{i=1}^n \frac{n+1-2i}{2} e_i \}$. Recall the element $w_0 \in S_n$ $(w_0(i) = n+1-i)$. We have $w_0 \rho = -\rho$ so $w_0 \cdot \lambda = w_0 (\lambda + \rho) - \rho = w_0 \lambda - 2 \rho = -\lambda$.

Recall that to $\mu \in \Lambda$ we can assign the following representations of $\mathfrak{g}_\mu$: the Verma module $\Delta(\mu)$ & its irreducible quotient $L(\mu)$. For $\mu = w \cdot \lambda$, the only irreducibles that can occur in $\text{ker} [\Delta(w \cdot \lambda) \to L(w \cdot \lambda)]$ are $L(u \cdot \lambda)$ where $u \cdot \lambda < w \cdot \lambda$, Sec 1.1 in Lec 16. If we know their multiplicities, we can express the (unknown) $\text{ch} L(w \cdot \lambda)$ via (known) $\text{ch} \Delta(u \cdot \lambda)$ when $u \cdot \lambda < w \cdot \lambda$.

**Thm** (Kazhdan-Lusztig conjecture (1979) proved by Beilinson-Bernstein & Brylinski-Kashiwara (1981), reproved a number of times afterwards).

- The multiplicity of $L(u \cdot \lambda)$ in $\Delta(w \cdot \lambda)$ is $c_{u,w}(1) \Rightarrow \text{ch} \Delta(w \cdot \lambda) = \sum_{u \leq w} c_{u,w}(1) \text{ch} (L(u \cdot \lambda))$.

- $\text{ch} L(w \cdot \lambda) = \sum_{u \leq w} (-1)^{\ell(w) - \ell(u)} c_{w,u}(1) \text{ch} (\Delta(u \cdot \lambda))$.

Note that the upper triangularity in the theorem is different from what we had before; it’s stronger, as $u < w \Rightarrow u \cdot \lambda < w \cdot \lambda$ (exercise).

2) Complements

Kazhdan-Lusztig bases/polynomials are remarkable objects that were extensively studied since they were discovered. Yet, much is still unknown. Here’s a brief account of some developments.
2.1) Properties of KL polynomials.

- Positivity: Theorem in Sec. 1.4, in particular, means that $c_{u,w}(1) \geq 0 \not\equiv u \in W$. More is true: $c_{u,w} \in \mathbb{Z}_{\geq 0}$. This is completely not obvious from the construction in Sec. 1.3—or any other combinatorial construction. The claim that $c_{u,w} \in \mathbb{Z}_{\geq 0}$ was proved by Kazhdan and Lusztig in 1980: they checked that the coefficients of $c_{u,w}$ are the dimensions of stalks of $IC(BuB/B, Q)$ on $BuB$. A connection to the IC's (Intersection complexes) is an important ingredient of the classical proofs of the theorem.

No enumerative meanings of the coefficients of $c_{u,w}$ (or of $c_{w,w}(1)$) is known (in general) — and none is expected to exist. Still KL combinatorics has deep connections to the classical enumerative combinatorics (for example, via the theory of "cells").

\[ \mathbb{Z}_{\geq 0} \text{ if } w \not\equiv u \]

- Other restrictions: one has $c_w(u) \equiv 2^{\ell(w) - \ell(u)} P_{w,u}(u^*)$ for $P_{w,u}$, a polynomial $w$ integral coefficients. One can trace this from the definition. Another restriction is that the $P_{w,u}(0) = 1$. And that's it: any polynomial $w$ non-negative integer coefficients & constant term 1 arises as $P_{w,u}$ for some $w,u \in S_n$ for some $n$. P. Polo, "Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups", Representation theory, 1999.

- Kazhdan-Lusztig inversion formula: Theorem in Sec. 1.4 implies

\[ \sum_{y \in W} (-1)^{|w| - |y|} c_{uy}(1) c_{y_0,w_0}(1) = S_{u,w} \not\equiv u, w \in W. \]
we have $\sum_{y \in W} (-1)^{\ell(w) - \ell(y)} c_{y, y} c_{y, 0, w_0} = S_{w, w}$.

This is a combinatorial shadow of a deep representation theoretic fact: the principal block of category $O$ is Koszul self-dual. We'll mention some more on this later.

- **Combinatorial invariance conjecture**

A fundamental issue in computing $c_{w, u}$ is that to compute them one needs to start with $w = 1$ and do induction on the Bruhat order. At the same time, there's a lot of evidence suggesting that $c_{w, u}$ depends not on $w, u$ themselves but on the interval between $w$ and $u$ in the Bruhat graph (the full subgraph, whose vertices are all vertices on a path from $w$ to $u$). For example, the interval between $s_1, s_2, s_4$, and $s_2$ in the Bruhat graph looks like

![Bruhat graph](image)

and each time the interval between $w$ and $u$ is $\wrightarrow u$ we should have $c_{w, u} = w^3$. The general conjecture is known as the "combinatorial invariance conjecture", see [arXiv:2111.15161](https://arxiv.org/abs/2111.15161) for recent developments and more details.

### 2.2) Generalizations

Recall that in Section 2.2 of Lec 20 we have defined a Coxeter group $W$ with generators $s_i, i \in I$, and relations $s_i^2 = 1$
\[(S_is_j)^{m_j} = 1\] for a collection of elements \(m_j \in \mathbb{Z}_{\geq 1} \cup \{\infty\}\)

If we fix a collection of indeterminates \(t_i, s.t. t_i = t_j\) if \(s_i, s_j\) are conjugate (in terms of the \(m_j\)'s this boils down to the condition that \(t_i = t_j\) as long as \(m_j\) is odd (and \(< \infty\)), then we can define the generic Hecke algebra \(H_{(t_i)}(W)\) (Section 2.3 of Lec 20). We can consider its equal parameter specialization \(\mathcal{H}_0(W) = \mathbb{Z}[v^\pm] \otimes_{\mathbb{Z}[t_i]} H_{(t_i)}(W)\) \(w.t. t_i \mapsto v^\pm\).

We can still consider the bar-involution \(\overline{\cdot}\) and define the Kazhdan-Lusztig basis \(C_w, w \in W\), as in the \(S_n\)-case. The resulting basis and the corresponding KL polynomials share many similarities to the \(S_n\)-case, as the same geometric picture holds (where one replaces the usual flag variety \(G/B\) for \(G = S_n\) with the flag variety for the corresponding Kac-Moody group). For example, we still have \(C_{wu} \in \mathbb{N}_{\geq 0}[v]\).

For the general Coxeter groups, the flag varieties aren't there. Relatively recently, Elias and Williamson, "Hodge theory of Soergel bimodules", Ann. Math. (2) 181 (2014), 1089-1136, proved that \(C_{wu} \in \mathbb{N}_{\geq 0}[v]\) for all Coxeter groups. The proof is algebraic but heavily uses geometric insights (it emulates the Hodge theory from Algebraic geometry).

More generally, we can specialize \(t_i\)'s to different powers of \(v\) (when we have more than one conjugacy class, \(W(B_2)\) a.k.a. the order 8 dihedral group, is the simplest example). Here we can
specialize $t_i$'s to different powers of $i$ (usually, both negative & even). But if the powers are not the same, the positivity may fail - already in type $B_2$. 