Hecke algebra/category, part V.

1) Category $\mathcal{O}$.
2) Projective objects.
3) Complements: variants and relatives of $\mathcal{O}$.

1)

Kazhdan-Lusztig conjecture concerns the behavior of Verma modules and their irreducible quotients. However, the classical proofs (those from 81 mentioned in Sec. 1.4 of Lec 21 as well as the alternative proof of Soergel) require understanding a certain ambient category, the BGG (Bernstein-I. Gelfand-S. Gelfand) category $\mathcal{O}$ of $\mathfrak{g}$-modules introduced by the three authors in 1976. This category is what we are going to study next - and it's a close relative of the "the Hecke category."

1.1) Definition & examples of objects.

Recall the Lie algebra $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{C})$, its Cartan subalgebra $\mathfrak{h}$ (the diagonal matrices), the positive roots $\varepsilon_i-\varepsilon_j\in\mathfrak{h}^*$ $(1\leqslant i<j\leqslant n)$, the weight lattice $\Lambda=\{\sum_{i=1}^n \lambda_i\varepsilon_i \mid \lambda_i\in\mathbb{Z}\}/\mathbb{Z}(\varepsilon_1+\ldots+\varepsilon_n)$ and the order $\leqslant$ on Λ:
$\lambda\leqslant\mu \Leftrightarrow \mu-\lambda\in \operatorname{Span}_{\mathbb{Z}_{\geqslant 0}}(\text{positive roots})$.

Definition: The category $\mathcal{O}$ is defined as follows. The objects are all $\mathfrak{g}$-modules M satisfying the following conditions:

(i) M is a weight module: $M=\bigoplus_{\mu\in\Lambda} M_\mu$, where $\mathfrak{h}$ acts by

μ on M_μ & $\dim M_\mu < \infty$ $\forall \mu \in \Lambda$.

(ii) The weights of M are bounded from above: $\exists\, \lambda_1, \ldots \lambda_k \in \Lambda$ s.t. $\forall\, \mu \in \Lambda$ w. $M_\mu \neq \{0\}$ $\exists\, i$ w. $\mu \leq \lambda_i$.

(iii) M is finitely generated over $\mathcal{U}(\mathfrak{g})$.

The morphisms in $\mathcal{O}$ = the $\mathfrak{g}$-linear maps (i.e. $\mathcal{O}$ is a "full subcategory" of $\mathfrak{g}$-Mod).

Examples (of objects): • $\Delta(\lambda) \in \mathcal{O}$ $\forall\, \lambda \in \Lambda$.

• if $M \in \mathcal{O}$, then so is any quotient of M. Below we'll see the same is true for subs but so far it's not clear why (iii) holds.

• if $M_1, M_2 \in \mathcal{O}$, then $M_1 \oplus M_2 \in \mathcal{O}$.

• Suppose $M \in \mathcal{O}$ & V is a finite dimensional $\mathfrak{g}$-module. We claim that $V \otimes M \,(= V \otimes_{\mathbb{C}} M) \in \mathcal{O}$:

(i): part 1 of Prob 3 in HW3

(ii): exercise.

(iii): follows from the next lemma

Lemma: if $M_0 \subset M$ is a finite dimensional subspace s.t. $\mathcal{U}(\mathfrak{g}) M_0 = M$, then $\mathcal{U}(\mathfrak{g})(V \otimes M_0) = V \otimes M$.

Proof: Recall $\mathcal{U}(\mathfrak{g})_{\leq i} := \operatorname{Span}_{\mathbb{C}}(\xi_1 \ldots \xi_j \mid j \leq i \;\&\; \xi_1, \ldots, \xi_j \in \mathfrak{g})$. Set $M_{\leq i} :=$ $= \mathcal{U}(\mathfrak{g})_{\leq i} M_0$ so that $M = \bigcup_i M_{\leq i}$. It's enough to check that

$V\otimes_{\mathbb{C}} M_{\leq i} \subset U(\mathfrak{g})_{\leq i}(V\otimes M_0)$. The proof is by induction with the base, $i=0$, being vacuous. Pick $\xi_1 \dots \xi_i \in \mathfrak{g}$, $m_0\in M_0$, $v\in V$. We note

$$\xi_1\dots\xi_i(v\otimes m_0) = [\xi_i(v\otimes m_0) = \underbrace{\xi_i v}_{\in V}\otimes m_0 + v\otimes \xi_i m_0] \in v\otimes \xi_1\dots\xi_i m_0 + V\otimes M_{\leq i-1}.$$

By the inductive assumption, we are done. $\square$

1.2) Infinitesimal blocks & finite length.

First, let's discuss the decomposition into infinitesimal blocks.

The Weyl group $W=S_n$ acts on $\mathfrak{h}^*$ & Λ by $w\cdot\lambda = w(\lambda+\rho)-\rho$. Let $\mathfrak{h}^*/W$ Λ/W denote the set of orbits. Since every $M\in\mathcal{O}$ is a weight module, it decomposes as $M=\bigoplus_{\chi\in\mathfrak{h}^*/W} M^\chi$, where on M^χ, $\forall m\in M^\chi$, $z\in$ center of $U(\mathfrak{g})$ we have $(z-HC_z(\chi))^k m=0$ for some k, see HW3.

Exercise: if $\chi\neq\chi'\in\mathfrak{h}^*/W$, then $\mathrm{Hom}_{\mathfrak{g}}(M^\chi, N^{\chi'})=0$ $\forall M,N\in\mathcal{O}$.

Definition: for $\chi\in\mathfrak{h}^*/W$, define the infinitesimal block $\mathcal{O}^\chi$ of $\mathcal{O}$ as the full subcategory of all objects $M\in\mathcal{O}$ w. $M=M^\chi$.

Example: $\Delta(\lambda)\in\mathcal{O}^\chi$ w. $\chi=W\cdot\lambda$.

Proposition: $\mathcal{O}=\bigoplus_{\chi\in\mathfrak{h}^*/W}\mathcal{O}^\chi$, which explicitly means the following.

1) $\forall M\in\mathcal{O}$, we have $M=\bigoplus_\chi M^\chi$ (already know) & $M^\chi=\{0\}$ for all but finitely many $\chi\in\mathfrak{h}^*/W$.

2) For $M,N\in\mathcal{O}$, we have $\mathrm{Hom}_{\mathcal{O}}(M,N)=\bigoplus_\chi \mathrm{Hom}_{\mathcal{O}^\chi}(M^\chi,N^\chi)$.

Proof: 1) Observe that any direct sum of $\mathcal{U}(\mathfrak{g})$-modules which is finitely generated must be finite. Applying this to $M = \bigoplus_{\chi} M^{\chi}$ and using (iii) of Def'n in Sec 1.1, we get 1). 2) is exercise. $\square$

Corollary: Every object in $\mathcal{O}$ has finite length (i.e admits a JH filtration).

Proof: We know that M is a finite direct sum of M^{χ}'s, so it's enough to show that each M^{χ} has finite length. This is Proposition in Sec 1.1 of Lec 16. $\square$

Exercise: 1) Show that every finite length $\mathfrak{g}$-module is finitely generated over $\mathcal{U}(\mathfrak{g})$. Deduce that, for $M\in\mathcal{O}$, every $\mathfrak{g}$-submodule of M is also in $\mathcal{O}$.

2) Show that $\nabla(\lambda) = (\mathbb{D}\Delta(\lambda)$, see Prob 3 in HW3) is in $\mathcal{O}$ (and $\mathbb{D}$ preserves $\mathcal{O}$).

1.3) Irreducible objects

Proposition: The set $\operatorname{Irr}(\mathcal{O})$ of isomorphism classes of irreps in $\mathcal{O}$ is in bijection w. Λ via $\lambda \mapsto L(\lambda)$. This restricts to $\mathcal{X} \xrightarrow{\sim} \operatorname{Irr}(\mathcal{O}^{\chi})$.

Proof: Let $L\in \operatorname{Irr}(\mathcal{O})$. Its weights are bounded from the above, so there's a highest weight, say $\lambda \rightsquigarrow$ nonzero homomorphism $\Delta(\lambda)\to L$. Since $\Delta(\lambda)$ has the unique irreducible quotient, $L(\lambda)$, we must have $L\simeq L(\lambda)$. Since $L(\lambda)\not\simeq L(\mu)$ for $\lambda\neq\mu$ (different highest weights), this finishes the proof. The last claim is left as an exercise. $\square$

Corollary: $\mathcal{O}^\chi \neq \{0\} \iff \chi \in \Lambda/W$

Proof: $\Leftarrow$: $L(\lambda) \in \mathcal{O}^\chi$ for $\lambda \in \chi$ (by Example in Sec 1.2)

$\Rightarrow$: every object in $\mathcal{O}^\chi$ has finite length so if $\mathcal{O}^\chi \neq \{0\}$, then there's an irreducible object in $\mathcal{O}^\chi$. But the irreducibles in $\mathcal{O}$ are $L(\lambda)$'s and $L(\lambda) \in \mathcal{O}^{W\cdot\lambda}$ $\square$

2) Projective objects

Recall that $P \in \mathcal{O}$ (or $\mathcal{O}^\chi$) is projective if $\operatorname{Hom}(P,\bullet)$ is an exact functor $\mathcal{O}$ (resp. $\mathcal{O}^\chi$) $\longrightarrow \mathbb{C}$-Vect.

Theorem: Suppose χ is a free orbit ($\Leftrightarrow$ no stabilizer in W; in fact, this assumption can be removed). The category $\mathcal{O}^\chi$ has enough projectives meaning every object is a quotient of some projective object.

Since every object in $\mathcal{O}^\chi$ has finite length this is equivalent to the formally weaker claim that $\forall\, L \in \operatorname{Irr}(\mathcal{O})\ \exists$ projective P_L w. $P_L \twoheadrightarrow L$. The proof of this equivalence is left as an exercise.

To prove the theorem (in the next lecture) we will find one projective object and then cook more by applying certain functors to it.

Proposition: Let $\lambda \in \Lambda$ be such that $\lambda+\rho$ is dominant. Then $\Delta(\lambda)$ is projective in $\mathcal{O}$ and in $\mathcal{O}^\chi$, where $\chi = W\cdot\lambda$.

Proof: Step 1: We claim that $\lambda > w\cdot\lambda$ if $w\cdot\lambda \neq \lambda$. Namely, otherwise we can find $w \in W \setminus \{1\}$ s.t. $\mu = w(\lambda+\rho)$ is maximal in $W(\lambda+\rho)$. But $s_i\mu = \mu - \langle \mu, h_i \rangle \alpha_i$ and so $\mu \geq s_i\mu \Leftrightarrow \langle \mu, h_i\rangle \geq 0$ so μ must be dominant. But $\lambda+\rho$ is the only dominant weight in $W(\lambda+\rho)$.

Step 2: We claim that if $M \in \mathcal{O}^{\chi}$ and $M_\mu \neq 0$ for some $\mu \in \Lambda$, then $\mu \leq \lambda$. Take a JH filtration $\{0\} = M_0 \subset M_1 \subseteq \ldots \subset M_k = M$. If $M_\mu \neq 0$, then $(M_i/M_{i-1})_\mu \neq 0$ for some i. So we can assume M is simple $\Rightarrow$ [Proposition in Sec 1.3] $M = L(\nu)$ for some $\nu \in \Lambda$. But $L(\nu) \in \mathcal{O}^{\chi} \Rightarrow \nu \in \chi$. So $M_\mu \neq 0 \Rightarrow \exists\, w \in W \mid \mu \leq w\cdot\lambda \leq [\text{Step 1}] \leq \lambda$, finishing this step.

Step 3: Now we can prove that $\Delta(\lambda)$ is projective in $\mathcal{O}^{\chi}$. We have $\mathrm{Hom}_{\mathcal{O}^{\chi}}(\Delta(\lambda), M) \xrightarrow{\sim} \{m \in M_\lambda \mid e_\alpha m = 0 \ \forall \text{ positive roots}\} = [\text{by Step 2, } M_{\lambda+\alpha} = \{0\}] = M_\lambda$. But $M \mapsto M_\lambda$ is an exact functor, so $\Delta(\lambda)$ is projective in $\mathcal{O}^{\chi}$.

Step 4: $\Delta(\lambda)$ is projective in $\mathcal{O}$. Since $\Delta(\lambda) \in \mathcal{O}^{\chi}$, by Prop'n in Sec 1.2, $\mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^{\chi}}(\Delta(\lambda), M^{\chi})$.

Exercise: $\bullet^{\chi}: \mathcal{O} \to \mathcal{O}^{\chi}$ is an exact functor.

Now we are done by Step 3 (the composition of exact functors is exact) $\square$

3) Complements

Just as the KL polynomials are a "golden standard" for multiplicities of irreducible representations in "standard" representations, the infinitesimal block $\mathcal{O}^0$ (and its direct analogs for more general Kac-Moody algebras, especially finite dimensional and affine) is a "golden standard" of a category in Representation theory - meaning that many other categories can be understood by comparing them to $\mathcal{O}^0$ and it's "variants."

3.1) Variants:

- Singular blocks: If X is a free W-orbit, then $\mathcal{O}^X$ and $\mathcal{O}^0$ are equivalent, they are known as regular blocks. The categories $\mathcal{O}^X$, where X is not free, are known as singular blocks. They are "simpler" then the regular ones (e.g. by Prob 1 in HW3, $\mathcal{O}^{-\rho}$ is equivalent to $\mathbb{C}$-Vect) but play an important role, in particular, in understanding the regular blocks. Many aspects of their study can be reduced to regular blocks. We will elaborate on this in the next lecture.

- Parabolic categories $\mathcal{O}$: Consider $G = SL_n(\mathbb{C})$ and its Borel subgroup B (of all upper triangular matrices). Let $M \in \mathcal{O}$.

Exercise: Every $m \in M$ is contained in a finite dimensional $\mathfrak{b}$-subrepresentation.

Now pick a decomposition $n=n_1+\dots+n_k$ into the sum of positive integers. Consider the subalgebra $\mathfrak{p}\subset\mathfrak{g}$ of block upper triangular matrices w. blocks of sizes $n_1,\dots n_k$. E.g. for $n=1+\dots+1$ we get $\mathfrak{p}=\mathfrak{b}$, for $n=n$ we get $\mathfrak{p}=\mathfrak{g}$, while for $3=1+2$ and $3=2+1$, we get $\mathfrak{p}=\left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\}$ and $\mathfrak{p}=\left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}\right\}$. Such $\mathfrak{p}$ is known as a parabolic subalgebra.

Definition: the parabolic category $\mathcal{O}$ for $\mathfrak{p}$, denoted by $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory in $\mathcal{O}$ consisting of all objects M s.t every $m\in M$ is contained in a finite dimensional $\mathfrak{p}$-subrepresentation.

By the previous exercise, $\mathcal{O}^{\mathfrak{b}}=\mathcal{O}$. And $\mathcal{O}^{\mathfrak{g}}$ is the category of finite dimensional $\mathfrak{g}$-reps, exercise.

Let's give an example of an object in $\mathcal{O}^{\mathfrak{p}}$.

Example: Let $\mathfrak{l}\subset\mathfrak{p}$ be the subalgebra of all block diagonal matrices (e.g. for $3=2+1$, we get $\mathfrak{l}=\left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}\right\}$). We have the natural projection $\mathfrak{p}\twoheadrightarrow\mathfrak{l}$. Pick $\lambda\in\Lambda$ s.t. $\langle\lambda,h_i\rangle\geq 0$ for all i w. $e_i\in\mathfrak{l}$ (e.g. for $3=2+1$, there is only one condition: $\langle\lambda,h_1\rangle\geq 0$. Then we can form the finite dimensional irreducible representation $L_{\mathfrak{l}}(\lambda)$ of $\mathfrak{l}$ w. highest weight λ (here we take the order where we only use α w. $e_\alpha\in\mathfrak{l}$). It's the tensor product of irreducible representations of $\mathfrak{sl}_{n_1},\dots,\mathfrak{sl}_{n_k}$ w. appropriate highest weights and the center of $\mathfrak{l}$ acting by scalars so that $\mathfrak{h}$ acts on the highest weight space by λ.

Thx to $\mathfrak{p} \twoheadrightarrow \mathfrak{l}$, we can view $L_{\mathfrak{l}}(\lambda)$ as a $\mathfrak{p}$-module. Then we form the parabolic Verma module $\Delta^{\mathfrak{p}}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} L_{\mathfrak{l}}(\lambda)$.

Exercise: $\Delta^{\mathfrak{p}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}$. Moreover, $\mathrm{Hom}_{\mathfrak{g}}(\Delta^{\mathfrak{p}}(\lambda), M) = \mathrm{Hom}_{\mathfrak{p}}(L_{\mathfrak{l}}(\lambda), M)$.

Parabolic category $\mathcal{O}^{\mathfrak{p}}$ has other similarities w. $\mathcal{O}$: "highest weight structure", decomposition into infinitesimal blocks...

• Affine category $\mathcal{O}$: this is the analog of category $\mathcal{O}$ for an affine Kac-Moody algebra $\hat{\mathfrak{sl}}_n$ (or $\tilde{\mathfrak{sl}}_n$, see the complement to Lec 20). Here we fix the "level", κ, = the scalar for the action of the central element c and consider modules M w. weight decomposition $M = \bigoplus_{\lambda} M_{\lambda}$, where we sum over $\lambda \in \hat{\mathfrak{h}}^*$ w. $\langle \lambda, c\rangle = \kappa$ & $\langle \lambda, h_i \rangle \in \mathbb{Z}$ for $i = 1, \ldots, n-1$. The definition of the category $\mathcal{O}$ goes through in this setting. But the situation is richer and more complicated. Depending on κ, the category $\mathcal{O}$ functions in four different modes:

- $\kappa \notin \mathbb{Q}$
- $\kappa \in -n + \mathbb{Q}_{<0}$ (negative level)
- $\kappa \in -n + \mathbb{Q}_{>0}$ (positive level)
- $\kappa = -n$ (critical level)

In the first (easy) case we just get the direct sum of several copies of $\mathcal{O}$. In the last three cases, we get new (and more complicated) categories that are "governed" by the affine Weyl group $W(\tilde{A}_n)$.

3.2) Relatives. There's a bunch of categories of interest for Representation theory that look quite different from category $\mathcal{O}$ (or its variants) but nevertheless are related to them. This relation often allows to compute suitably understood characters in these categories.

(a) Modules over cyclotomic degenerate affine Hecke algebras. Consider the degenerate affine Hecke algebra $\mathcal{H}(d)$. Fix an unordered ℓ-tuple $\mathcal{X}=(\mathcal{X}_1,\dots,\mathcal{X}_\ell)$ of complex numbers. The most interesting case is when they are integers, which is what we are going to assume. Consider the quotient $\mathcal{H}_{\mathcal{X}}(d)=\mathcal{H}(d)/(\prod_{i=1}^{\ell}(X_1-\mathcal{X}_i))$ – this is the algebra in the title. It's category of modules is realized as a "quotient category" of the direct sums of some blocks of $\mathcal{O}^{\mathfrak{p}}$ for suitable n & $\mathfrak{p}$: J. Brundan, A. Kleshchev "Schur-Weyl duality for higher levels", Selecta Math, 2008.

"Quotient category" roughly means that we force some irreducible objects to be zero. We can describe which irreducibles are made 0 when we pass from $\mathcal{O}^{\mathfrak{p}}$ to $\mathcal{H}_{\mathcal{X}}(d)$. So we get a combinatorial classification of irreducible $\mathcal{H}_{\mathcal{X}}(d)$-modules as well as their character formulas (in a suitable sense). As every irreducible $\mathcal{H}(d)$-module factors through some $\mathcal{H}_{\mathcal{X}}(d)$ ($\mathcal{X}$ may fail to be an integer, it may also fail to be unique), this also leads to the classification and computation of characters of irreducible $\mathcal{H}(d)$-modules, the problem that was mentioned in Remark 4.7, [RT1].

All other – more interesting – examples have to do with the affine type. They include:

(b) The category of finite dimensional representations of "Lusztig's form of $U_q(\mathfrak{sl}_n)$," where q is a root of unity. Kazhdan & Lusztig proved that this category is equivalent to the parabolic category $\mathcal{O}$ for $\hat{\mathfrak{sl}}_n^\kappa$, where the parabolic subalgebra is $\mathfrak{sl}_n \otimes \mathbb{C}[t] + \mathbb{C}c$ and the level κ is negative ($\in -n + \mathbb{Q}_{<0}$) w. $q = e^{\pi\sqrt{-1}\kappa}$. This is the series of four papers referenced for Lec 17. The approach of Kazhdan-Lusztig was inspired by developments in Math Physics. Later an alternative approach, more in the framework of the geometric Representation theory was found in S. Arkhipov, R. Bezrukavnikov, V. Ginzburg, "Quantum groups, the loop Grassmanian, and the Springer resolution, J. Amer. Math. Soc. 17 (2004).

(c) Representations of semisimple algebraic groups & their Lie algebras in characteristic $p \gg 0$. The Steinberg tensor product theorem reduces the case of groups to the case of Lie algebras. Recall that the elements of the form $x^p - x^{[p]} \in \mathcal{U}(\mathfrak{g})$ for $x \in \mathfrak{g}$ are central. They span a copy of $\mathfrak{g}$ in $\mathcal{U}(\mathfrak{g})$, compare to Sec 1 of Lec 10. So to $\chi \in \mathfrak{g}^*$ we can form the quotient $\mathcal{U}^\chi$ of dimension $p^{\dim \mathfrak{g}}$ (compare to Complement to Lec 9).

We first look at $\mathcal{U}^0$. By Problem 4 in HW 2, every rational representation of G viewed as a $\mathcal{U}(\mathfrak{g})$-module factors through $\mathcal{U}^0$. $\mathcal{U}^0$-modules coming from rational G-reps inherit weight decompositions.

The action of $\mathcal{U}^\circ$ is compatible w. the weight decomposition. Namely, $\mathfrak{g}$ is Λ-graded (w. $\mathfrak{h}$ in deg 0, e_α in deg α, f_α in deg $-\alpha$). This is a Lie algebra grading: $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. It gives rise to an algebra grading on $\mathcal{U}(\mathfrak{g})$. For $x \in \mathfrak{g}_\lambda$, we have $x^{[p]} \in \mathfrak{g}_{p\lambda}$ & $x^p \in \mathcal{U}(\mathfrak{g})_{p\lambda}$. In particular, the relations of $\mathcal{U}^\circ$ are Λ-homogeneous so $\mathcal{U}^\circ$ inherits a grading: $\mathcal{U}^\circ = \bigoplus_{\lambda \in \Lambda} \mathcal{U}^\circ_\lambda$.

Definition: By a weight $\mathcal{U}^\circ$-module we mean a finite dimensional $\mathcal{U}^\circ$-module V together w. decomposition $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$ s.t. $\mathcal{U}^\circ_\mu V_\lambda \subset V_{\lambda+\mu}$ $\forall \lambda, \mu \in \Lambda$.

An example of a weight module is a baby Verma module $\underline{\Delta}^\circ(\lambda)$ (generalizing what we've seen in Sec 3 of Lec 9, to give a definition is an exercise). Each $\underline{\Delta}^\circ(\lambda)$ has a unique irreducible quotient $L^\circ(\lambda)$ and our problem is to compute the multiplicities of $L^\circ(\lambda)$ in $\underline{\Delta}^\circ(\lambda)$. This problem was first solved by Andersen-Jantzen-Sorgel, see references for Lec 7. They proved that for $p \gg 0$ these multiplicities are the same as the analogous multiplicities for the analog of this category for the quantum groups <u>over $\mathbb{C}$</u>, where we choose $\sqrt[p]{1}$ for q.

A relationship between the category of weight modules on the quantum group side and the affine category $\mathcal{O}$ is that the former is a "limit version" of the latter, in a suitable sense.

We can ask about the case of more general χ. Here we can reduce the study of $\mathcal{U}^\chi$-mod to the case when χ is nilpotent. Here's what we know in chronological order (for general semisimple Lie algebras - this is one of the situations when the case of $\mathfrak{sl}_n$ is relatively simple, but, say, $\mathfrak{so}_n$ is hard enough). Suppose $p \gg 0$.

• R. Bezrukavnikov, I. Mirkovic, D. Rumynin, "Localization of modules for a simple Lie algebra in prime characteristic", Ann. Math. (2), 167 (2008): counted the number of irreducible representations of $\mathcal{U}^\chi$. For example, the K_0 (see Lecture 14.5) of the principal block is identified w. the homology of an algebraic variety called the Springer fiber.

• R. Bezrukavnikov, I. Mirkovic "Representation of semisimple Lie algebras in prime characteristic and noncommutative Springer resolution", Ann. Math. 178 (2013). This paper identifies the basis of the classes of irreducible modules in the K_0 of the principal block of $\mathcal{U}^\chi$ w. a certain basis in essentially the aforementioned homology proving a conjecture of Lusztig.

• R. Bezrukavnikov, I. Losev "Dimensions of modular irreducible representations of semisimple Lie algebras" arXiv: 2005.10030

We tweak $\mathcal{U}^\chi$-mod "slightly", in essence considering some kind of weight modules w. - and this is a tweak - an additional finite group action. The resulting category turns out to be closely related to the affine category $\mathcal{O}$, somewhat similarly to the case $\chi=0$.

The tweak we use seems relatively innocent, e.g. for classical

Lie algebras the finite group in question is $(\mathbb{Z}/2\mathbb{Z})^{\oplus ?}$ - easy! Nevertheless many aspects of the usual category $\mathcal{U}^\chi$-mod remain mysterious starting with basic ones: for $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$ (as was mentioned, the $\mathfrak{sl}_n$-case is easy, e.g. the finite group is trivial) give a combinatorial classification of $\mathcal{U}^\chi$-irreps.

(d) Representations of the Hecke algebra $\mathcal{H}_\epsilon(S_n)$, where ϵ is a root of 1 of order $\leq n$. This reduces to (b) via a quantum version of the Schur-Weyl duality.

(e) And more!