Hecke algebra/category. Part VI

1) Projective objects, cont'd.

1.0) Recap/goal:

Last time we have introduced category $\mathcal{O}=\bigoplus_{\mathcal{X}\in\Lambda/W}\mathcal{O}^{\mathcal{X}}$. The irreducible objects in $\mathcal{O}$ (resp. $\mathcal{O}^{\mathcal{X}}$) are precisely the $L(\lambda)$'s w. $\lambda\in\Lambda$ (resp. $\lambda\in\mathcal{X}$). In particular, if $\mathcal{X}$ is free (the most interesting case), it contains a unique dominant weight (exercise), say λ. Then $W\xrightarrow{\sim}\operatorname{Irr}(\mathcal{O}^{\mathcal{X}})$ via $w\mapsto L(w\cdot\lambda)$. See Sec 1.3 of Lec 22 for proofs.

We have stated the following theorem in Sec 2 of Lec 22.

Theorem: Let $\mathcal{X}$ be free (for simplicity). Then $\mathcal{O}^{\mathcal{X}}$ has enough projectives: every (irreducible) object is a quotient of a projective.

We have seen that $\Delta(\lambda)$ is projective, Prop'n in Sec 2 of Lec 22.

Goal: • for $i=1,\dots,n-1$, define a reflection functor $\Theta_i:\mathcal{O}^{\mathcal{X}}\to\mathcal{O}^{\mathcal{X}}$.
For $w=s_{i_1}\dots s_{i_\ell}$ ($\ell=\ell(w)$) show that $\Theta_{i_\ell}\dots\Theta_{i_1}\Delta(\lambda)$ is projective w. epimorphism onto $\Delta(w\cdot\lambda)$, hence $L(w\cdot\lambda)$.

Accomplishing this will prove the theorem.

1.1) Biadjoint functors and projectives.

Pick $\mathcal{X}_1,\mathcal{X}_2\in\Lambda/W$. For additive functors $\mathcal{F}:\mathcal{O}^{\mathcal{X}_1}\to\mathcal{O}^{\mathcal{X}_2}$, $\mathcal{G}:\mathcal{O}^{\mathcal{X}_2}\to\mathcal{O}^{\mathcal{X}_1}$, we say that $\mathcal{F}$ & $\mathcal{G}$ are biadjoint if we have natural (functorial in M_1, M_2) isomorphisms of <u>abelian groups</u>.

$$\mathrm{Hom}_{\mathcal{O}^{\chi_1}}(M_1, \mathcal{G}M_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^{\chi_2}}(\mathcal{F}M_1, M_2),\ \mathrm{Hom}_{\mathcal{O}^{\chi_2}}(M_2, \mathcal{F}M_1) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^{\chi_1}}(\mathcal{G}M_2, M_1)$$

The reason why we care is as follows, Prob 2 in HW6 in 380, F21.

Fact: biadjoint $\mathcal{F}$ & $\mathcal{G}$ are exact and send projectives to projectives.

Here is our main example.

Proposition: Let V be a finite dimensional $\mathfrak{g}$-rep. Then

$$\mathcal{F} := (V\otimes \bullet)^{\chi_2}: \mathcal{O}^{\chi_1} \to \mathcal{O}^{\chi_2},\ \mathcal{G} := (V^*\otimes \bullet)^{\chi_1}: \mathcal{O}^{\chi_2} \to \mathcal{O}^{\chi_1}$$

are biadjoint. taking generalized eigenspace for Z-action

Proof: Let $M_i \in \mathcal{O}^{\chi_i}$. Then

$\mathrm{Hom}_{\mathcal{O}^{\chi_2}}((V\otimes M_1)^{\chi_2}, M_2) \xrightarrow{\sim}$ [Prop. in Sec 1.2 of Lec 22]

$\mathrm{Hom}_{\mathcal{O}}(V\otimes M_1, M_2) \xrightarrow{\sim}$ [Prob 2.2 in HW3]

$\mathrm{Hom}_{\mathcal{O}}(M_1, V^*\otimes M_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}^{\chi}}(M_1, (V^*\otimes M_2)^{\chi_1})$.

The other adjunction is proved in the same way. □

1.2) Translation functors.

These are an important special case of functors in the proposition. Pick $\chi_i \in \Lambda/W$ $(i=1,2)$. $\exists!\ \lambda_i \in \chi_i$ s.t. $\lambda_i + \rho \in \Lambda_+$. Further, let μ denote the unique dominant element in $W(\lambda_2-\lambda_1)$.

Definition: $T_{\lambda_1\to\lambda_2} := (L(\mu)\otimes\bullet)^{\chi_2}: \mathcal{O}^{\chi_1}\to\mathcal{O}^{\chi_2}$ is translation functor

A justification for this choice of μ: in many cases we can control what the functor does to Verma modules. We need two special cases.

Proposition: Suppose χ_1 is free. Then
(a) $T_{\lambda_1\to\lambda_2}(\Delta(w\cdot\lambda_1)) = \Delta(w\cdot\lambda_2)$, $\forall\ w\in W$.
(b) $T_{\lambda_2\to\lambda_1}(\Delta(w\cdot\lambda_2))$ has a filtration by $\Delta(wu\cdot\lambda_1)$ where u runs over $\mathrm{Stab}_{W,\cdot}(\lambda_2) = \{u\in W \mid u\cdot\lambda_2=\lambda_2\}$; each $\Delta(wu\cdot\lambda_1)$ for such u occurs once.

Example (of b)): i) $\lambda_1=0$, $\lambda_2=-\rho$: this is Prob. 4.1 in HW3.

Proof of Proposition: We'll prove (b) — (a) is similar but easier, we leave it as an exercise. By Prob 2 in HW3, $T_{\lambda_2\to\lambda_1}\Delta(w\cdot\lambda_2) = (L(\mu)\otimes\Delta(w\cdot\lambda_2))^{\chi_1}$ has a filtration by Vermas as follows: we take all weights $\mu_1,\dots,\mu_k$ of $L(\mu)$ (with multiplicities) s.t.

$$w\cdot\lambda_2+\mu_i\in W\cdot\lambda_1$$

& $\mu_i>\mu_j \Rightarrow i<j$ and then get a filtration

$$\{0\}=M_0\subset M_1\subset\ldots\subset M_k = (L(\mu)\otimes\Delta(w\cdot\lambda_2))^{\chi_1} \text{ w. } M_i/M_{i-1}\simeq\Delta(w\cdot\lambda_2+\mu_i)$$

To show that only $\Delta(wu\cdot\lambda_1)$ appear we need to check that:

$\mu_i = v(\lambda_1+\rho)-w(\lambda_2+\rho)$ for some $v\in W \Leftrightarrow \mu_i = wu(\lambda_1-\lambda_2)$ w. $u\in\mathrm{Stab}_{W,\cdot}(\lambda_2)$.

Each $\Delta(wu\cdot\lambda_1)$ must then appear once b/c $\dim L(\mu)_{x\mu}=1\ \forall x\in W$.

To prove "$\Leftarrow$" we identify $\mathfrak{h}^*$ w. $\{\sum_{i=1}^n x_i\varepsilon_i \mid \sum_{i=1}^n x_i=0\}$ and consider the bilinear form $(x,y)=\sum_{i=1}^n x_iy_i$ on $\mathfrak{h}^*$. It's W-invariant. Set $|x|^2=(x,x)$. It's enough to check

$$|\mu_i|^2\leq|\mu|^2, \text{ w. equality} \Leftrightarrow \mu_i\in W\mu. \tag{1}$$

$$|v(\lambda_1+\rho)-w(\lambda_2+\rho)|^2\geq|\mu|^2, \text{ w. equality} \Leftrightarrow v\cdot\lambda_2=w\cdot\lambda_2. \tag{2}$$

Proof of (1): can assume μ_i is dominant by replacing it w. a W-conjugate. Since $\mu_i\leq\mu \Rightarrow (\mu_i,\mu-\mu_i)\geq 0 \Rightarrow |\mu|^2\geq|\mu_i|^2+|\mu_i-\mu|^2$. This shows (1).

Proof of (2): This reduces to showing $(\lambda_1+\rho, \lambda_2+\rho-w(\lambda_2+\rho))\geq 0$ w. equality iff the 2nd argument is 0. Note that $\lambda_2+\rho\geq w(\lambda_2+\rho)$ so

$$\lambda_2+\rho-w(\lambda_2+\rho)=\sum_{j=1}^{n-1}m_j\alpha_j \quad \text{w. } m_j\geq 0$$

On the other hand the entries of $\lambda_1+\rho$ are strictly decreasing $\Rightarrow$ $(\lambda_1+\rho,\alpha_j)>0\ \forall j=1,\dots n-1$. This proves (2) $\square$

Remarks: 1) The proof shows $(L(\mu')\otimes\Delta(w\cdot\lambda_2))^{\chi_1}\neq 0 \Rightarrow |\mu'|^2\geq|\mu|^2$. This is another justification for our choice of μ.

2) Consider special case: $\mathrm{Stab}_{W_\bullet}(\lambda_2)=\{1,s_i\}$, i.e., $(\lambda+\rho)_j\geq(\lambda+\rho)_{j+1}$ w. equality iff $i=j$. (6) tells us we have a SES

$$0\to\Delta(w'\cdot\lambda_1)\longrightarrow T_{\lambda_2\to\lambda_1}\Delta(w\cdot\lambda_2)\longrightarrow\Delta(w''\cdot\lambda_1)\longrightarrow 0,$$

where $\{w',w''\}=\{w,ws_i\}$. By the properties of the filtration of $L(\mu)\otimes\Delta(w\cdot\lambda_2)$ recalled in the proof, we have $w''\cdot\lambda_1<w'\cdot\lambda_1$. Since

$w''=w's_i$ we have $w''\cdot\lambda_1 = w'\cdot\lambda_1 - ((\lambda_1+\rho)_i - (\lambda_1+\rho)_{i+1})(\varepsilon_{w'(i)} - \varepsilon_{w'(i+1)})$.
So $w''\cdot\lambda_1 < w'\cdot\lambda_1 \Leftrightarrow w'(i) < w'(i+1) \Leftrightarrow \ell(w') < \ell(w'')$.

1.3) Reflection functors

Pick $\lambda \in \Lambda_+ \Rightarrow \mathrm{Stab}_{W,\cdot}(\lambda) = \{1\}$. Further for $i=1,\dots,n-1$, pick $\lambda_i \in \Lambda \mid \lambda_i+\rho \in \Lambda_+$ w. $\mathrm{Stab}_{W,\cdot}(\lambda_i) = \{1, s_i\}$. Define the reflection (a.k.a. wall-crossing) functor $\Theta_i := T_{\lambda_i\to\lambda}\circ T_{\lambda\to\lambda_i} : \mathcal{O}^X \to \mathcal{O}^X$ ($X = W\cdot\lambda$).

The following proposition accomplishes the goal in Section 1.0 and so completes the proof of the theorem.

Proposition: Let $w = s_{i_1}\dots s_{i_\ell}$ w. $\ell = \ell(w)$. Then $\Theta_{i_\ell}\dots\Theta_{i_1}\Delta(\lambda)$ is a projective object that surjects onto $\Delta(w\cdot\lambda)$.

Proof: Induction on ℓ. The base, $\ell=0$, follows b/c $\Delta(\lambda)$ is projective. For the induction step, we observe that each $T_{?\to ?}$ admits a biadjoint by Proposition in Sec 1.1, hence sends projectives to projectives by Fact there. The claim that Θ_{i_ℓ} sends projectives to projectives follows. Also, Θ_{i_ℓ} is exact, as a composition of exact functors. Apply Θ_{i_ℓ} to $\Theta_{i_{\ell-1}}\dots\Theta_{i_1}\Delta(\lambda) \twoheadrightarrow \Delta(\underbrace{ws_{i_\ell}}_{s_{i_1}\dots s_{i_{\ell-1}}}\cdot\lambda)$ getting $\Theta_{i_\ell}\dots\Theta_{i_1}\Delta(\lambda) \twoheadrightarrow \Theta_{i_\ell}\Delta(ws_{i_\ell}\cdot\lambda)$

By a) of Proposition in Sec 1.2, have $T_{\lambda\to\lambda_i}\Delta(ws_{i_\ell}\cdot\lambda) = \Delta(w\cdot\lambda_{i_\ell})$. So $\Theta_{i_\ell}\Delta(ws_{i_\ell}\lambda) = T_{\lambda_{i_\ell}\to\lambda}\Delta(w\cdot\lambda_{i_\ell})$. By Remark 2 in Sec 1.2, have SES

$$0 \to \Delta(ws_{i_\ell}\cdot\lambda) \to \Theta_{i_\ell}\Delta(ws_{i_\ell}\cdot\lambda) \to \Delta(w\cdot\lambda) \to 0$$

So $\Theta_{i_\ell}\dots\Theta_{i_1}\Delta(\lambda) \twoheadrightarrow \Delta(w\cdot\lambda)$ and we are done $\square$

Example: $\mathfrak{g}=\mathfrak{sl}_2$: $\lambda_1=-1$. Take $\lambda=0$. Then $\Theta_1\Delta(\lambda)=\mathbb{C}^2\otimes\Delta(-1)$, compare to Problem 4 in HW3.

1.4) Comments

1) To prove the theorem, an easier argument will do, see [H3], Sec. 3.8. However the endofunctors Θ_i are very important and will be used later as well.

2) Here's another application of translation functors: if $\chi_1,\chi_2\in\Lambda/W$ s.t. the unique elements $\lambda_i\in\chi_i$ w. $\lambda_i+\rho\in\Lambda_+$ satisfy $\mathrm{Stab}_{W_\bullet}(\lambda_1)=$ $=\mathrm{Stab}_{W_\bullet}(\lambda_2)$, then $T_{\lambda_1\to\lambda_2}: \mathcal{O}^{\chi_1}\leftrightarrows\mathcal{O}^{\chi_2}: T_{\lambda_2\leftarrow\lambda_1}$ are mutually quasiinverse category equivalences, [H3], Sec 7.8. We care mostly about the case where the orbit is trivial. The equivalence above allows us to restrict to the case when $\lambda=0$ (principal block).

3) Let $W\cdot\lambda_1$ be free. Then one can show ([H3], Secs 7.9) that $T_{\lambda_1\to\lambda_2}(L(w\cdot\lambda_1))=L(w\cdot\lambda_2)$ if $\ell(wu)\geqslant\ell(w)$ $\forall\, u\in\mathrm{Stab}_{W_\bullet}(\lambda_2)$ and 0 else. This allows to reduce the computation of multiplicities of irreps in Vermas to the case of principal block $\mathcal{O}^0$.

1.5) Endomorphisms of a projective generator.

Let $\chi\in\Lambda/W$ be free. By a projective generator in $\mathcal{O}^\chi$ we mean a projective object P s.t $\mathrm{Hom}_{\mathcal{O}^\chi}(P,L)\neq 0$ $\forall\, L\in\mathrm{Irr}(\mathcal{O}^\chi)$. Since $\mathrm{Irr}(\mathcal{O}^\chi)$ is finite, Theorem in Sec 1.0 shows that a projective generator exists.

Set $A := \mathrm{End}_{\mathcal{O}^\chi}(P)^{opp}$. This is a finite dimensional algebra: recall that all objects in $\mathcal{O}^\chi$ have finite length (Sec 1.2 of Lec 22) so $\dim \mathrm{Hom}_{\mathcal{O}^\chi}(M,N)<\infty\ \forall M,N\in\mathcal{O}_\chi$ (exercise).

Here's one reason to care about A. Notice that $\forall M\in\mathcal{O}^\chi$, the vector space $\mathrm{Hom}_{\mathcal{O}^\chi}(P,M)$ carries a natural A-module structure - by composition - and so $\mathrm{Hom}_{\mathcal{O}^\chi}(P,\cdot)$ can be viewed as a functor $\mathcal{O}^\chi\to A\text{-mod}$, the category of finite dimensional A-modules.

Proposition: The functor $\mathrm{Hom}_{\mathcal{O}^\chi}(P,\cdot):\mathcal{O}^\chi\to A\text{-mod}$ is a category equivalence (w. quasi-inverse $P\otimes_A\cdot$)

The proof is based on three facts: the $\mathcal{O}^\chi$ has finitely many irreducibles, P is a projective generator, and all objects in $\mathcal{O}^\chi$ have finite length, see [E], Section 6.3.

Example: $\mathfrak{g}=\mathfrak{sl}_2$, $\lambda=0$. Then, by Example in Sec 1.3, we can take $P=\Delta(0)\oplus\mathbb{C}^2\otimes\Delta(-1)$. We have SES $0\to\Delta(0)\to P\to\Delta(-2)\to 0$. Note that $\mathrm{Hom}_{\mathcal{O}^0}(\Delta(0),\Delta(-2))=0$, while $\mathrm{Hom}_{\mathcal{O}^0}(\Delta(-2),\Delta(0))\simeq\mathbb{C}$, a nonzero element is an inclusion $\Delta(-2)\hookrightarrow\Delta(0)$.

Let $\varepsilon_0,\varepsilon_{-2}$ denote the identity elements in $\mathrm{End}(\Delta(0))$, $\mathrm{End}(\mathbb{C}^2\otimes\Delta(-1))$ viewed as elements of $\mathrm{End}(P)$, they are idempotents. Also consider the elements $\alpha:\Delta(0)\hookrightarrow\mathbb{C}^2\otimes\Delta(-1)$ and $\beta:\mathbb{C}^2\otimes\Delta(-1)\twoheadrightarrow\Delta(-2)\hookrightarrow\Delta(0)$, again viewed as elements of $\mathrm{End}(P)$. Note that $\gamma=\alpha\beta\neq 0$.

Exercise: The elements $\varepsilon_0, \varepsilon_{-2}, \alpha, \beta, \gamma$ form a basis in $\mathrm{End}(P)$ and the product is recovered from

$\varepsilon_0 + \varepsilon_{-2} = 1$, $\varepsilon_0 \alpha = \alpha \varepsilon_{-2} = \beta \varepsilon_0 = \varepsilon_{-2} \beta = \beta \alpha = 0$ ($\Rightarrow \alpha^2 = \beta^2 = 0$, $\alpha \varepsilon_0 = \alpha$, etc.).