Hecke algebra/category-, part VII

1) Projectives, finished.
2) (Two out of) three theorems of Soergel.
3) Complements.

1.0) Recap. Let \( X \) be a free \( W \)-orbit for the \( \cdot \)-action of \( W \) on the weight lattice \( \Lambda \) and \( \lambda \in X \cap \Lambda^+ \). We have produced certain projective objects in \( \mathcal{O}^X \). Namely, let \( w = (s_{i_1}, \ldots, s_{i_e}) \) be a reduced expression for \( w \in W \), meaning that \( w = s_{i_1} \cdots s_{i_e} \) & \( \ell = \ell(w) \). Set \( \mathcal{O}_w = \mathcal{O}_{i_e} \cdots \mathcal{O}_{i_1} \), an exact endofunctor of \( \mathcal{O}^X \) sending projectives to projectives. We have seen that \( \mathcal{O}_w \Delta(\lambda) \to \Delta(w \cdot \lambda) \), we used this to show that \( \mathcal{O}^X \) has enough projectives.

1.1) Category of projectives. Let \( \mathcal{O}^X \)-proj denote the full subcategory of \( \mathcal{O}^X \) consisting of projective objects. The following theorem describes the objects of \( \mathcal{O}^X \)-proj.

**Thm:** 1) \( \exists \, \lambda' \in X \) \( \exists ! \) projective \( P(\lambda') \in \mathcal{O}^X \) s.t. \( \dim \text{Hom}_{\mathcal{O}^X}(P(\lambda'), \mathcal{O}(\mu)) = \delta_{\lambda\mu} \).
2) \( P \in \mathcal{O}^X \)-proj, we have \( P \cong \bigoplus_{\lambda \in X} P(\lambda) \otimes d_{\lambda} \) w. \( d_{\lambda} = \dim \text{Hom}_{\mathcal{O}^X}(P, \mathcal{O}(\lambda)) \).

To prove this let’s discuss decompositions into \( \bigoplus \) of indecomposables.

Let \( R \) be a \( C \)-algebra. We say \( M \in R\text{-mod} \) is indecomposable if \( M \ncong M_1 \oplus M_2 \) for \( M_1, M_2 \in R\text{-mod}, \) nonzero.
Assume now
\[ \dim \text{End}_R(M) < \infty \tag{*} \]

**Lemma:** TFAE

1) $M$ is indecomposable
2) \( \forall \alpha \in \text{End}_R(M) \exists \lambda \in C, m > 0 \text{ s.t. } (\alpha - \lambda)^m = 0 \)
3) $\text{End}_R(M) = C1 \oplus \text{rad} \text{End}_R(M)$.

**Proof - exercise.**

**Proposition:** Let $M \in R$-mod satisfy (\(\ast\))
1) $M$ decomposes as $\bigoplus_{k=1}^n M_k$, where $M_k$ is indecomposable.
2) Moreover, if $M = \bigoplus_{k=1}^n M_k'$ is another such decomposition, then $k = l$ and $M_k \cong M_k'$ for some $G \in S_k$ (*Krull-Schmidt property*).

1) is an exercise. 2) is the *Krull-Schmidt* theorem, [E], Section 3.8.

**Sketch of proof of Thm:**

- Existence of $P(\lambda')$: in Sec 1.5 of Lec 23, we have established an equivalence $O^A \to \text{A-mod}$ for a finite dimensional algebra $A$. It's enough to establish an analogous result in $\text{A-mod}$. Let $L \in \text{Irr}(A)$.

Choose a primitive idempotent $\mathbf{e} \in \text{End}(L) \subseteq A/\text{rad} A$. We can find $\mathbf{e} \in A$ w. $\mathbf{e} + \text{rad} A = \mathbf{e} \& \mathbf{e}^2 = e_1$ ([E], Sec 8.1). Then $P_1 = A\mathbf{e}$ is projective and $\dim \text{Hom}_A(A\mathbf{e}, L^{'}) = \dim \mathbf{e} L^{' \in S_1}$. \(\forall \mathbf{L} \in \text{Irr}(A)\).
The remaining statements: Let \( P \in \proj(A) \) w. nonzero homomorphism \( \varphi: P \to L \). Let \( \varphi: P \to L \) be a nonzero homomorphism. We can find \( \tilde{\varphi}: P \to P \), \( \tilde{\varphi}: P \to P \) making the following commutative:

\[
\begin{array}{ccc}
P & \xrightarrow{\tilde{\varphi}} & P \\
\varphi & \downarrow & \varphi \\
L & \xrightarrow{\tilde{\varphi}} & L
\end{array}
\]

Consider \( \tau = \tilde{\varphi} \tilde{\varphi} \in \text{End}(P) \).

**Exercise:** Use 2) of Lemma to show \( \tau \) is invertible.

- Deduce that \( P = \ker \tilde{\varphi} \oplus \text{im} \tilde{\varphi} \).
- Complete the proof. \( \square \)

**Example:** i) \( P(\lambda) = \Delta(\lambda) \) - the r.h.s. is projective & indecomposable.

ii) Consider the object \( \mathcal{J}_{\lambda} \Delta(-p) = (\lambda(\lambda+p) \otimes \Delta(p))^x \). This object is projective b/c \( \Delta(-p) \) is & \( \mathcal{J}_{\lambda} \) sends projectives to projectives (Secs 1.1.1.2 of Lec 23). Similarly to Prob 4 in HW3, \( \mathcal{J}_{\lambda} \Delta(-p) \to \Delta(w_{0} \cdot \lambda) \) (for \( \lambda = 0 \), get \( w_{0} \cdot 0 = -2p \)). It follows that \( P(w_{0} \cdot \lambda) \) is a direct summand in \( \mathcal{J}_{\lambda} \Delta(-p) \). In fact, \( \mathcal{J}_{\lambda} \Delta(-p) \) is indecomposable so \( \mathcal{J}_{\lambda} \Delta(-p) = P(w_{0} \cdot \lambda) \) - we'll elaborate on this in the next lecture.

**Exercise:** \( \dim \text{Hom}_{\mathfrak{g} \mathfrak{x}}(P(\lambda), M) = \text{multiplicity of } \Delta(\lambda) \text{ in } M \) (hint: induct on the length of \( \mathcal{J} \mathfrak{g} \mathfrak{x} \) filtration of \( M \) using that \( \text{Hom}_{\mathfrak{g} \mathfrak{x}}(P_{\cdot}, \cdot) \) is exact).
1.2) Verma filtrations on projectives.

By a Verma filtration on $M \in \mathcal{O}^X$ we mean a filtration

$$0 = M_0 < M_1 < \ldots < M_k = M \text{ s.t. } M_i/M_{i-1} \text{ is a Verma.}$$

Example: $\mathcal{O}_w \Delta(\lambda)$ has a Verma filtration: $\mathcal{O}_w$ is exact & have SES

$$0 \rightarrow \Delta(u^i \cdot \lambda) \rightarrow \bigoplus_{i} \Delta(u^i \cdot \lambda) \rightarrow \Delta(u^i \cdot \lambda) \rightarrow 0, \forall u \in W,$

where \( \{u^i, u^j\} = \{u, u^i; \frac{1}{2}(\ell(u^i) - \ell(u^j))\} \), see the proof of Prop in Sec 1.3 of Lec 23.

So the successive quotients of $\mathcal{O}_w \Delta(\lambda)$ (i.e. of them) are labelled by subwords of $W$, for the subword $u$, the corresponding subquotient is $\Delta(u \cdot \lambda)$, where $u$ is a (not necessarily reduced) expression for $u$. All claims in this paragraph are proved by induction on $\ell$ (exercise).

$M$ can have different Verma filtrations but they all have the same successive quotients up to permutation. This follows from the following claim (see the complement section for a discussion).

Fact 1: Let $M \in \mathcal{O}$ be Verma filtered: $0 = M_0 < M_1 < \ldots < M_k$. Then

$$\# \{i \mid M_i/M_{i-1} \cong \Delta(\mu)\} = \dim \overline{\text{Hom}}(M_i, \nu(\mu)), \forall \mu \in \Lambda.$$ 

Now we turn to the indecomposable projectives $P(\mu), \mu \in X$.

Theorem: 1) For all $\mu \in X$, $P(\mu)$ admits a Verma filtration.

2) For all $\lambda \in X$, the multiplicity of $\Delta(\lambda)$ in $P(\mu)$ coincides w the multiplicity of $L(\mu)$ in $\Delta(\lambda)$ (BGG reciprocity).
Sketch of proof: Using Thm in Sec 1.1, we see that $P(w \cdot \lambda)$ is a direct summand of $\bigoplus_w \Delta(\lambda)$. Now 1) follows from:

**Fact 2.** Let $M_1, M_2 \in \mathfrak{O}_X$. If $M_1 \oplus M_2$ admits a Verma filtration, then so do $M_1, M_2$.

We'll prove this in the complement section.

To prove (2) we notice that the multiplicity of $\mathcal{L}(\mu)$ in $\Delta(\gamma)$ coincides w. that in $\mathcal{V}(\gamma)$ b/c $\mathcal{V}(\gamma) = \mathcal{T} \Delta(\gamma)$ & $\mathcal{L}(\mu) = \mathcal{T} \mathcal{L}(\mu) + f_\gamma$ (Prob. 3 in HW3). Then

mult of $\Delta(\gamma)$ in $P(\mu) = [\text{Fact 1}] = \dim \operatorname{Hom}_{\mathfrak{O}_X}(P(\mu), \mathcal{V}(\gamma)) = 
\begin{align*}
\text{[Last exer. in Sec 1.1]} & = \text{multiplicity of } \mathcal{L}(\mu) \text{ in } \mathcal{V}(\gamma), \text{ equiv. in } \Delta(\gamma) \quad \square 
\end{align*}

1.3) Decomposing $\bigoplus_w \Delta(\lambda)$.

Let's discuss the decomposition of $\bigoplus_w \Delta(\lambda)$ into $\bigoplus$ of indecomposables and why we should care. From Example in Sec. 1.2 we know that $\Delta(w \cdot \lambda)$ occurs in the Verma filtration of $\bigoplus_w \Delta(\lambda)$ once—and as a quotient—for all other $\Delta(u \cdot \lambda)$ that occur satisfy the condition:

(*): $u$ is equal to a proper subword of $s_{i_1} \cdots s_{i_k}$.

Combinatorial fact: (*), $\Leftrightarrow u < w$ (in Bruhat order, Sec 1.3 in Lec 21).

Exercise: Deduce that
• $\text{Hom}_\omega(\bigoplus U \Delta(\lambda), L(u; \lambda)) \neq 0 \Rightarrow u \leq w \& \text{ for } u = w, \dim = 1 \text{ (hint: look at } \text{Homs from successive filtration quotients)}.

• $\bigoplus U \Delta(\lambda) = P(w; \lambda) \oplus \bigoplus_{u \leq w} P(u; \lambda)^{m_{u,w}}$ for some $m_{u,w} \in \mathbb{N}_{\geq 0}$.

If we know $m_{u,w}$'s we can compute the multiplicities of $\Delta(u; \lambda)$'s in $P(w; \lambda)$ recursively. By Thm in Sec 1.2, this is the multiplicity of $C(w; \lambda)$ in $\Delta(u; \lambda)$ - which is what we want to compute starting Lec 16.

2) (Two out of 3) Three theorems of Soergel


• Computation of $\text{End}(\mathcal{T}_{p \to \lambda} \Delta(p))$. To compute the endomorphism of a projective generator - or even most $P(y)$'s - is hard. But for $\mathcal{T}_{p \to \lambda} \Delta(p)$ ($= P(w; \lambda)$), Example in Sec 1.1) the endomorphism algebra turns out to be a very classical object.

Let $m_0 = \{ f \in \mathbb{C}[y^*]^W \mid f(0) = 0 \}$, a maximal ideal. Consider the algebra of coinvariants $\mathbb{C}[ y^* ]^W = \mathbb{C}[ y^* ]/ \mathbb{C}[y^*]m_0$. It has dimension $|W|/|e(\mathbb{C}[y^*])|$ is a free $\mathbb{C}[y^*]^W$-module of rank $|W|$. We have seen (Prob 4.3 of HW 3) that $\dim \text{End}(\mathcal{T}_{p \to \lambda} \Delta(p)) = |W|$ as well.

**Theorem 1:** $\text{End}_\omega(\mathcal{T}_{p \to \lambda} \Delta(p)) \cong \mathbb{C}[y^*]^W$. 

6
Functor $V$ Consider the functor [endomorphisms of $T_{\rightarrow \Lambda(-p)}$]

\[ V : \text{Hom}_{\mathcal{O}^x}(T_{\rightarrow \Lambda(-p)}, \cdot) : \mathcal{O}^x \rightarrow \mathbb{C}[y^*]_{\text{coW}} - \text{mod} \]

It's exact; $T_{\rightarrow \Lambda(-p)} = \text{P}(\mathcal{O}^x)$ and $V((\mathcal{O}^x)) = \mathbb{C}_{\text{coW}}^j w_0$

Since $V$ kills all irreps but one, it looks like this functor loses a lot of information and isn't going to be useful in our study of $\mathcal{O}^x$. However, we have:

**Theorem 2**: $V$ is fully faithful on $\mathcal{O}^x$-proj (i.e. preserves Hom's).

What Theorem 2 tells us is that to describe $\mathcal{O}^x$-proj, it's enough to understand its image in $\mathbb{C}[y^*]_{\text{coW}}$-mod. The image turns out to be (the ungraded version) of the category of Soergel modules to be discussed next time.

3) Complements

Here we provide proofs of 2 facts mentioned in Sec 1.2.

**Fact 1**: this follows from the claim, Prob. 3.7 in HW3, that

\[ \text{Ext}^i(\Delta(\mu), \Delta(\nu)) = 0 \quad \text{the Ext is in } \mathcal{O}^x \], $\forall \mu, \nu \in \Lambda$,

compare to solution of Prob 4.3 in HW3.

**Fact 2**: We will use the following claim similar to Prob 5.2 in HW2:

if SES in $\mathcal{O}$, $0 \rightarrow \Delta(\mu) \rightarrow M \rightarrow \Delta(\nu) \rightarrow 0$, doesn't split, then $\mu > \nu$.

We'll also use that $\text{Hom}_{\mathcal{O}^x}(\Delta(\mu), \Delta(\nu)) = 0$ if $\mu > \nu$ and $\dim \text{End}_{\mathcal{O}^x}(\Delta(\mu)) = 1$. 


The proof is by induction on the length of the filtration. Let \( \nu \) be a maximal weight of a Verma in the filtration of \( M_1 \oplus M_2 \).

Thanks to the previous paragraph, "\( \Delta(\nu) \) slides to the bottom of the filtration" so we have a SES:

\[
0 \to \Delta(\nu) \oplus k \to M_1 \oplus M_2 \to N \to 0
\]

where \( N \) is filtered by other Vermas. Note that since \( N \) is filtered by Vermas with highest weights \( \neq \nu \), \( \text{Hom}_{\mathbb{C}}(\Delta(\nu), N) = 0 \) (from the left exactness) of \( \text{Hom} \). So \( k = \dim \text{Hom}_{\mathbb{C}}(\Delta(\nu), M_1 \oplus M_2) \). Also observe that any nonzero homomorphism \( \Delta(\nu) \to M_1 \oplus M_2 \) - because every homomorphism factors through \( \Delta(\nu) \to \Delta(\nu) \oplus k \) & \( \text{End}_{\mathbb{C}}(\Delta(\nu)) = \mathbb{C} 1 \). Since

\[
\text{Hom}_{\mathbb{C}}(\Delta(\nu), M_1 \oplus M_2) = \text{Hom}_{\mathbb{C}}(\Delta(\nu), M_1) \oplus \text{Hom}_{\mathbb{C}}(\Delta(\nu), M_2)
\]

Pick a nonzero element in one of the summands, say the first. It gives an embedding \( \Delta(\nu) \to M_1 \). Then we replace \( M_1 \) with \( M_1 / \text{im}(\Delta(\nu)) \) and proceed by induction.