Hecke algebra/category, part VII

1) Projectives, finished.
2) (Two out of) three theorems of Soergel.
3) Complements.

1.0) Recap. Let $\mathcal{X}$ be a free W-orbit for the $\cdot$-action of W on the weight lattice Λ and $\lambda\in\mathcal{X}\cap\Lambda_+$. We have produced certain projective objects in $\mathcal{O}^{\mathcal{X}}$. Namely, let $\underline{w}:=(s_{i_1},\dots,s_{i_\ell})$ be a reduced expression for $w\in W$, meaning that $w=s_{i_1}\dots s_{i_\ell}$ & $\ell=\ell(w)$. Set $\Theta_{\underline{w}}=\Theta_{i_\ell}\dots\Theta_{i_1}$, an exact endofunctor of $\mathcal{O}^{\mathcal{X}}$ sending projectives to projectives. We have seen that $\Theta_{\underline{w}}\Delta(\lambda)\twoheadrightarrow\Delta(w\cdot\lambda)$, we used this to show that $\mathcal{O}^{\mathcal{X}}$ has enough projectives.

1.1) Category of projectives. Let $\mathcal{O}^{\mathcal{X}}$-proj denote the full subcategory of $\mathcal{O}^{\mathcal{X}}$ consisting of projective objects. The following theorem describes the objects of $\mathcal{O}^{\mathcal{X}}$-proj.

Thm: 1) $\exists\ \lambda'\in\mathcal{X}\ \exists!$ projective $P(\lambda')\in\mathcal{O}^{\mathcal{X}}$ s.t. $\dim\operatorname{Hom}_{\mathcal{O}^{\mathcal{X}}}(P(\lambda'),L(\mu))=\delta_{\lambda'\mu}$.
2) $\forall P\in\mathcal{O}^{\mathcal{X}}$-proj, we have $P\simeq\bigoplus_{\lambda'\in\mathcal{X}}P(\lambda')^{\oplus d_{\lambda'}}$ w. $d_{\lambda'}=\dim\operatorname{Hom}_{\mathcal{O}^{\mathcal{X}}}(P,L(\lambda'))$.

To prove this let's disuss decompositions into $\oplus$ of indecomposables. Let R be a $\mathbb{C}$-algebra. We say $M\in R$-mod is *indecomposable* if $M\not\cong M_1\oplus M_2$ for $M_1,M_2\in R$-mod, nonzero.

Assume now

$$\dim_{\mathbb{C}} \operatorname{End}_R(M) < \infty \qquad (*)$$

Lemma: TFAE

1) M is indecomposable
2) $\forall \tau \in \operatorname{End}_R(M)\ \exists \alpha \in \mathbb{C}, m>0$ s.t. $(\tau-\alpha)^m = 0$
3) $\operatorname{End}_R(M) = \mathbb{C}1 \oplus \operatorname{rad} \operatorname{End}_R(M)$.

Proof – exercise.

Proposition: Let $M \in R$-mod satisfy $(*)$

1) M decomposes as $\bigoplus_{i=1}^{k} M_i$, where M_i is indecomposable.
2) Moreover, if $M = \bigoplus_{j=1}^{\ell} M_i'$ is another such decomposition, then $k=\ell$ & $M_i \xrightarrow{\sim} M_{\sigma(i)}'$ for some $\sigma \in S_k$ ("Krull-Schmidt property).

1) is an exercise. 2) is the Krull-Scmidt theorem, [E], Section 3.8.

Sketch of proof of Thm:

- Existence of $P(\lambda')$: in Sec 1.5 of Lec 23, we have established an equivalence $\mathcal{O}^\chi \xrightarrow{\sim} A$-mod for a finite dimensional algebra A. It's enough to establish an analogous result in A-mod. Let $L \in \operatorname{Irr}(A)$ Choose a primitive idempotent $\underline{\varepsilon} \in \operatorname{End}(L) \subseteq_{\oplus} A/\operatorname{rad} A$. We can find $\varepsilon \in A$ w. $\varepsilon_L + \operatorname{rad} A = \underline{\varepsilon}$ & $\varepsilon_L^2 = \varepsilon_L$ ([E], Sec. 8.1). Then $P_L = A\varepsilon_L$ is projective and $\dim \operatorname{Hom}_A(A\varepsilon, L') = \dim \varepsilon L' = \delta_{L,L'}\ \forall\ L' \in \operatorname{Irr}(A)$.

• The remaining statements: Let $P \in A\text{-proj} (\simeq \mathcal{O}^\chi\text{-proj})$ w. nonzero homomorphism $\varphi: P \twoheadrightarrow L$. Let $\psi: P_L \twoheadrightarrow L$ be a nonzero homomorphism. We can find $\tilde{\varphi}: P \to P_L$, $\tilde{\psi}: P_L \to P$ making the following commutative:

$$\begin{tikzcd} P \arrow[rr, "\tilde{\varphi}"] \arrow[dr, "\varphi"'] & & P_L \arrow[dl, "\psi"] \\ & L & \end{tikzcd} \qquad \begin{tikzcd} P_L \arrow[rr, "\tilde{\psi}"] \arrow[dr, "\psi"'] & & P \arrow[dl, "\varphi"] \\ & L & \end{tikzcd}$$

Consider $\tau = \tilde{\varphi}\tilde{\psi} \in \mathrm{End}(P_L)$.

Exercise: • Use 2) of Lemma to show τ is invertible.

• Deduce that $P \simeq \ker \tilde{\varphi} \oplus \underbrace{\operatorname{im} \tilde{\psi}}_{P_L}$

• Complete the proof. $\square$

Example: i) $P(\lambda) = \Delta(\lambda)$ - the r.h.s. is projective & indecomposable.

ii) Consider the object $\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho) = (L(\lambda+\rho)\otimes\Delta(-\rho))^\chi$. This object is projective b/c $\Delta(-\rho)$ is & $\mathcal{T}_{-\rho\to\lambda}$ sends projectives to projectives (Sec's 1.1, 1.2 of Lec 23). Similarly to Prob. 4 in HW3, $\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho) \twoheadrightarrow \Delta(w_0\cdot\lambda)$ (for $\lambda=0$, get $w_0\cdot 0 = -2\rho$). It follows that $P(w_0\cdot\lambda)$ is a direct summand in $\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho)$. In fact, $\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho)$ is indecomposable so $\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho) = P(w_0\cdot\lambda)$ - we'll elaborate on this in the next lecture.

Exercise: $\dim \mathrm{Hom}_{\mathcal{O}^\chi}(P(\lambda), M) =$ multiplicity of $L(\lambda)$ in M (hint: induct on the length of JH filtration of M using that $\mathrm{Hom}_{\mathcal{O}^\chi}(P,\cdot)$ is exact).

1.2) Verma filtrations on projectives.

By a Verma filtration on $M\in \mathcal{O}^\chi$ we mean a filtration $\{0\}=M_0\subset M_1\subset\ldots\subset M_k=M$ s.t. M_i/M_{i-1} is a Verma.

Example: $\Theta_{\underline{w}}\Delta(\lambda)$ has a Verma filtration: Θ_{s_i} is exact & have SES $0\to\Delta(u'\cdot\lambda)\to\Theta_i\Delta(u\cdot\lambda)\to\Delta(u''\cdot\lambda)\to 0$, $\forall u\in W$, where $\{u'',u'\}=\{u,us_i\}$ & $\ell(u')<\ell(u'')$, see the proof of Prop. in Sec 1.3 of Lec 23.

So, the succesive quotients of $\Theta_{\underline{w}}\Delta(\lambda)$ (2^ℓ of them) are labelled by sub-words of $\underline{w}$, for the subword $\underline{u}$, the corresponding subquotient is $\Delta(u\cdot\lambda)$, where $\underline{u}$ is a (not necessarily reduced) expression for u. All claims in this paragraph are proved by induction on ℓ (exercise).

M can have different Verma filtrations but they all have the same successive quotients up to permutation. This follows from the following claim (see the complement section for a discussion).

Fact 1: Let $M\in\mathcal{O}$ be Verma filtered: $\{0\}=M_0\subset M_1\subset\ldots\subset M_k$. Then
$$\#\{i\mid M_i/M_{i-1}\simeq\Delta(\mu)\}=\dim_{\mathbb{C}}\operatorname{Hom}(M,\nabla(\mu)),\ \forall\mu\in\Lambda.$$

Now we turn to the indecomposable projectives $P(\mu)$, $\mu\in X$.

Theorem: 1) For all $\mu\in X$, $P(\mu)$ admits a Verma filtration.
2) For all $\nu\in X$, the multiplicity of $\Delta(\nu)$ in $P(\mu)$ coincides w. the multiplicity of $L(\mu)$ in $\Delta(\nu)$ (BGG reciprocity).

Sketch of proof: Using Thm in Sec 1.1, we see that $P(w\cdot\lambda)$ is a direct summand of $\Theta_{\underline{w}}\Delta(\lambda)$. Now 1) follows from:

Fact 2: Let $M_1, M_2 \in \mathcal{O}^\chi$. If $M_1 \oplus M_2$ admits a Verma filtration, then so do M_1, M_2.

We'll prove this in the complement section.

To prove (2) we notice that the multiplicity of $L(\mu)$ in $\Delta(\nu)$ coincides w. that in $\nabla(\nu)$ b/c $\nabla(\nu) = \mathbb{D}\Delta(\nu)$ & $L(\mu) = \mathbb{D}L(\mu)$ $\forall \mu$ (Prob. 3 in HW3). Then

mult. of $\Delta(\nu)$ in $P(\mu)$ = [Fact 1] = $\dim \operatorname{Hom}_{\mathcal{O}^\chi}(P(\mu), \nabla(\nu))$ = [last exer. in Sec 1.1] = multiplicity of $L(\mu)$ in $\nabla(\nu)$, equiv. in $\Delta(\nu)$ $\square$

1.3) Decomposing $\Theta_{\underline{w}}\Delta(\lambda)$.

Let's discuss the decomposition of $\Theta_{\underline{w}}\Delta(\lambda)$ into $\oplus$ of indecomposables - and why we should care. From Example in Sec. 1.2 we know that $\Delta(w\cdot\lambda)$ occurs in the Verma filtration of $\Theta_{\underline{w}}\Delta(\lambda)$ once - and as a quotient - for all other $\Delta(u\cdot\lambda)$ that occur satisfy the condition:

(*) u is equal to a proper subword of $s_{i_1}\dots s_{i_\ell}$.

Combinatorial fact: (*) $\Leftrightarrow$ $u \prec w$ (in Bruhat order, Sec 1.3 in Lec 21).

Exercise: Deduce that

• $\mathrm{Hom}_{\mathcal{O}^\chi}(\Theta_{\underline{w}}\Delta(\lambda), L(u\cdot\lambda)) \neq 0 \Rightarrow u \preceq w$ & for $u=w$, $\dim = 1$ (hint: look at Homs from successive filtration quotients).

• $\Theta_{\underline{w}}\Delta(\lambda) = P(w\cdot\lambda) \oplus \bigoplus_{u \prec w} P(u\cdot\lambda)^{\oplus m_{u,w}}$ for some $m_{u,w} \in \mathbb{Z}_{\geq 0}$.

If we know $m_{u,w}$'s we can compute the multiplicities of $\Delta(u\cdot\lambda)$'s in $P(w\cdot\lambda)$ recursively. By Thm in Sec 1.2, this is the multiplicity of $L(w\cdot\lambda)$ in $\Delta(u\cdot\lambda)$ – which is what we want to compute starting Lec 16.

2) (Two out of) three theorems of Soergel

W. Soergel "Kategorie $\mathcal{O}$, Perverse Garben und Moduln über den Koinvarianten zur Weylgruppe", J. Amer. Math. Soc. 3 (1990).

• Computation of $\mathrm{End}(T_{-\rho\to\lambda}\Delta(-\rho))$. To compute the endomorphism of a projective generator – or even most $P(\mu)$'s – is hard. But for $T_{-\rho\to\lambda}\Delta(-\rho)$ ($= P(w_0\cdot\lambda)$, Example in Sec 1.1) the endomorphism algebra turns out to be a very classical object.

Let $\mathfrak{m}_0 = \{f \in \mathbb{C}[\mathfrak{h}^*]^W \mid f(0)=0\}$, a maximal ideal. Consider the algebra of "coinvariants" $\mathbb{C}[\mathfrak{h}^*]^{coW} = \mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]\mathfrak{m}_0$. It has dimension $|W|$ b/c $\mathbb{C}[\mathfrak{h}^*]$ is a free $\mathbb{C}[\mathfrak{h}^*]^W$-module of rk $|W|$. We have seen (Prob 4.3 of HW3) that $\dim \mathrm{End}(T_{-\rho\to\lambda}\Delta(-\rho)) = |W|$ as well.

Theorem 1: $\mathrm{End}_{\mathcal{O}^\chi}(T_{-\rho\to\lambda}\Delta(-\rho)) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^{coW}$.

• Functor $\mathbb{V}$: Consider the functor

endomorphisms of $\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho)$

$$\mathbb{V} := \mathrm{Hom}_{\mathcal{O}^\chi}(\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho), \cdot): \mathcal{O}^\chi \longrightarrow \overbrace{\mathbb{C}[\mathfrak{h}^*]^{coW}}\text{-mod}$$

It's exact; $\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho) = P(w_0\cdot\lambda) \Rightarrow \mathbb{V}(L(w\cdot\lambda)) = \mathbb{C}^{\delta_{w,w_0}}$.

Since $\mathbb{V}$ kills all irreps but one, it looks like this functor loses a lot of information and isn't going to be useful in our study of $\mathcal{O}^\chi$. However, we have:

Theorem 2: $\mathbb{V}$ is fully faithful on $\mathcal{O}^\chi$-proj (i.e preserves Hom's).

What Theorem 2 tells us is that to describe $\mathcal{O}^\chi$-proj, it's enough to understand its image in $\mathbb{C}[\mathfrak{h}^*]^{coW}$-mod. The image turns out to be (the ungraded version) of the category of Soergel modules to be discussed next time.

3) Complements.

Here we provide proofs of 2 facts mentioned in Sec 1.2.

Fact 1: this follows from the claim, Prob. 3.7 in HW3, that

$$\mathrm{Ext}^1(\Delta(\mu), \nabla(\nu)) = 0 \quad (\text{the Ext is in } \mathcal{O}^\chi), \ \forall \mu, \nu \in \Lambda,$$

compare to solution of Prob 4.3 in HW3.

Fact 2: We will use the following claim similar to Prob 5.2 in HW2: if SES in $\mathcal{O}$, $0 \to \Delta(\mu) \to M \to \Delta(\nu) \to 0$, doesn't split, then $\mu > \nu$. We'll also use that $\mathrm{Hom}_{\mathcal{O}}(\Delta(\mu), \Delta(\nu)) \neq 0 \Rightarrow \mu \leq \nu$ and $\dim \mathrm{End}_{\mathcal{O}^\chi}(\Delta(\mu)) = 1$.

The proof is by induction on the length of the filtration. Let λ be a maximal weight of a Verma in the filtration of $M_1 \oplus M_2$. Thanks to the previous paragraph, "$\Delta(\lambda)$ slides to the bottom of the filtration" so we have a SES:

$$0 \to \Delta(\lambda)^{\oplus k} \longrightarrow M_1 \oplus M_2 \longrightarrow N \longrightarrow 0$$

where N is filtered by other Vermas. Note that since N is filtered by Vermas w. highest weights $\not\leq \lambda$, $Hom_{\mathcal{O}x}(\Delta(\lambda), N) = 0$ (from the left exactness) of Hom). So $k = \dim Hom_{\mathcal{O}x}(\Delta(\lambda), M_1 \oplus M_2)$. Also observe that any nonzero homomorphism $\Delta(\lambda) \to M_1 \oplus M_2$ – because every homomorphism factors through $\Delta(\lambda) \longrightarrow \Delta(\lambda)^{\oplus k}$ & $End_{\mathcal{O}x}(\Delta(\lambda)) = \mathbb{C}1$. Since

$$Hom_{\mathcal{O}x}(\Delta(\lambda), M_1 \oplus M_2) = Hom_{\mathcal{O}x}(\Delta(\lambda), M_1) \oplus Hom_{\mathcal{O}x}(\Delta(\lambda), M_2)$$

Pick a nonzero element in one of the summands, say the first. It gives an embedding $\Delta(\lambda) \overset{\iota}{\hookrightarrow} M_1$. Then we replace M_1 w. $M_1/\iota(\Delta(\lambda))$ and proceed by induction.