Hecke algebra/category, Part VIII

1) Soergel (bi)modules.
2) Complements.

1.0) Recap: We consider the category $\mathcal{O}^\chi$ for $\chi = W\cdot\lambda$ w. $\lambda\in\Lambda_+$, free orbit. Inside, we consider the category $\mathcal{O}^\chi$-proj of projective objects. Every object uniquely decomposes as $\bigoplus$ of indecomposable projectives $P(w\cdot\lambda)$, $w\in W$. For example, $P(\lambda)=\Delta(\lambda)$, $P(w_0\cdot\lambda)=\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho)=(L(\lambda+\rho)\otimes\Delta(-\rho))^\chi$, Sec 1.1 of Lec 24.

For $i=1,\dots,n-1$, we have endofunctor $\Theta_i: \mathcal{O}^\chi\text{-proj}\to\mathcal{O}^\chi\text{-proj}$. For $\underline{w}=(s_{i_1}\dots s_{i_\ell})$, a reduced expression of $w\in W$, set $\Theta_{\underline{w}}=\Theta_{i_\ell}\dots\Theta_{i_1}$. Then (Sec 1.3 of Lec 24) we have $\Theta_{\underline{w}}\Delta(\lambda)=P(w\cdot\lambda)\oplus\bigoplus\limits_{u<w}P(u\cdot\lambda)^{\oplus ?}$. Bruhat order.

In Section 2 of Lec 24 we introduced the algebra $\mathbb{C}[\mathfrak{h}^*]^{coW}:=\mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]\mathfrak{m}_0$, $\mathfrak{m}_0:=\{f\in\mathbb{C}[\mathfrak{h}^*]^W \mid f(0)=0\}$. We have then considered the Soergel functor $\mathbb{V}$:

$$\mathbb{V}=\operatorname{Hom}_{\mathcal{O}^\chi}(\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho),\bullet): \mathcal{O}^\chi\longrightarrow\mathbb{C}[\mathfrak{h}^*]^{coW}\text{-mod}$$

We've stated that $\mathbb{V}$ is fully faithful on $\mathcal{O}^\chi$-proj and our task is to describe the image of $\mathbb{V}$. This is how the Soergel (bi)modules have first appeared. Since then they became a crucial tool in the geometric/categorical representation theory - and also useful for knot theory.

Exercise 1: Use the full faithfulness of $\mathbb{V}$ to show that $P\in\mathcal{O}^\chi$-proj is indecomposable $\Leftrightarrow$ $\mathbb{V}(P)\in\mathbb{C}[\mathfrak{h}^*]^{coW}$-mod is (hint: being indecomposable is about endomorphisms).

Example: $n=2$, $\lambda=0$. Then the indecomposable projectives (see Example in Sec 1.5 in Lec 23) are $\Delta(0)=P(0)$ & $\mathbb{C}^2\otimes\Delta(-1)=P(-2)$, includes into SES $0\to\Delta(0)\to\mathbb{C}^2\otimes\Delta(-1)\to\Delta(-2)\to 0$; $W=S_2 \curvearrowright \mathfrak{h}^*=\mathbb{C}$ by $\{\pm 1\}$
$\Rightarrow \mathbb{C}[\mathfrak{h}^*]^{coW}=\mathbb{C}[x]/(x^2)=[\text{Prob. 4.4 in HW3}]=\mathrm{End}_{\mathcal{O}^0}(\mathbb{C}^2\otimes\Delta(-1))$.
$\mathbb{V}(P(-2))=$ regular $\mathbb{C}[x]/(x^2)$-module
$\mathbb{V}(P(0))=\mathrm{Hom}_{\mathcal{O}^0}(P(-2),\Delta(0))=[$vector space of dim = mult. of $L(-2)=\Delta(-2)$ in $\Delta(0)]$ $=\mathbb{C}$, that has the unique $\mathbb{C}[x]/(x^2)$-module structure.

Exercise 2: Check $\mathbb{V}$ is fully faithful on $\mathcal{O}^0$-proj (hint: Sec 1.5 in Lec 23).

1.1) Im $\mathbb{V}$. Every indecomposable projective in $\mathcal{O}^\chi$ occurs as a direct summand in $\Theta_{\underline{w}}\Delta(\lambda)$. So, thx to exercise 1, we need to compute $\mathbb{V}(\Theta_{\underline{w}}\Delta(\lambda))$ and then decompose it into indecomposables.

We start by computing $\mathbb{V}(\Delta(\lambda))$. Note $\mathbb{C}[\mathfrak{h}^*]^{coW}$ is a local algebra, so it has the unique 1-dimensional module, to be denoted by $\mathbb{C}$.

Proposition: We have $\mathbb{V}(\Delta(\lambda))=\mathbb{C}$.

Proof: We claim that the functors $T_{\lambda_1\to\lambda_2}, T_{\lambda_2\to\lambda_1}$ (Sec 1.2 of Lec 23) are biadjoint. Indeed, let μ be the unique dominant weight in $W(\lambda_2-\lambda_1)$ so that $T_{\lambda_1\to\lambda_2}=(L(\mu)\otimes\bullet)^{\chi_2}$. Notice that $-\mu$ is the lowest weight of $L(\mu)^*$ - passing to the dual multiplies weights by -1. So the irrep in the definition of $T_{\lambda_2\to\lambda_1}$ is $L(\mu)^*$ and so $T_{\lambda_2\to\lambda_1}=(L(\mu)^*\otimes\bullet)^{\chi_1}$. By Prop'n in Sec 1.1 of Lec 25, $T_{\lambda_1\to\lambda_2}$ and $T_{\lambda_2\to\lambda_1}$ are biadjoint.

Now $\mathbb{V}(\Delta(\lambda)) = \mathrm{Hom}_{\mathcal{O}^\chi}(\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho), \Delta(\lambda)) = \mathrm{Hom}_{\mathcal{O}^{-\rho}}(\Delta(-\rho), \mathcal{T}_{\lambda\to-\rho}\Delta(\lambda))$
$= [$a) of Prop 1.2 in Sec 1.2 of Lec 25$] = \mathrm{Hom}_{\mathcal{O}^{-\rho}}(\Delta(-\rho), \Delta(-\rho)) = \mathbb{C}$ $\square$

Now we need to understand the interaction between $\mathbb{V}$ and Θ_i.

Note that a $\mathbb{C}[\mathfrak{h}^*]^{coW}$-module is the same thing as a $\mathbb{C}[\mathfrak{h}^*]$-module, where $\mathfrak{m}_0$ acts by 0. For $i=1,\dots n-1$, we write $\mathbb{C}[\mathfrak{h}^*]^{s_i}$ for the subalgebra of s_i-invariant elements. In particular, $\mathfrak{m}_0 \subset \mathbb{C}[\mathfrak{h}^*]^{s_i}$.

Exercise: Let $M \in \mathbb{C}[\mathfrak{h}^*]$-mod. If $\mathfrak{m}_0 M = \{0\}$, then $\mathfrak{m}_0(\mathbb{C}[\mathfrak{h}^*]\otimes_{\mathbb{C}[\mathfrak{h}^*]^{s_i}} M) = 0$.

So $\mathbb{C}[\mathfrak{h}^*]\otimes_{\mathbb{C}[\mathfrak{h}^*]^{s_i}} \bullet$ can be viewed as an endo-functor of $\mathbb{C}[\mathfrak{h}^*]^{coW}$-mod. And here's the third theorem of Soergel.

Theorem: We have a functor isomorphism $\mathbb{V}\circ\Theta_i \simeq \mathbb{C}[\mathfrak{h}^*]\otimes_{\mathbb{C}[\mathfrak{h}^*]^{s_i}} \mathbb{V}$.

Remark: For $i=1,\dots,n-1$, define the elementary Bott-Samelson $\mathbb{C}[\mathfrak{h}^*]$-bimodule BS_i as $\mathbb{C}[\mathfrak{h}^*]\otimes_{\mathbb{C}[\mathfrak{h}^*]^{s_i}}\mathbb{C}[\mathfrak{h}^*]$. So $\mathbb{C}[\mathfrak{h}^*]\otimes_{\mathbb{C}[\mathfrak{h}^*]^{s_i}} M = BS_i \otimes_{\mathbb{C}[\mathfrak{h}^*]} M$.

1.2) Graded (bi)modules.

Let R be a commutative $\mathbb{C}$-algebra equipped with a $\mathbb{Z}_{\geq 0}$-algebra grading: $R = \bigoplus_{i\geq 0} R_i$. A basic example, $R = \mathbb{C}[\mathfrak{h}^*]$, where $R_i = 0$ for odd i and for even i, R_i is the space of homogeneous polynomials of deg $i/2$.

By a graded R-module we mean an R-module M together w. vector space decomposition $M = \bigoplus_{j\in\mathbb{Z}} M_j$ s.t. $R_i M_j \subset M_{i+j}$ $\forall i,j$. A homomorphism of graded R-modules M, N is an R-linear map $\varphi: M \to N$ w. $\varphi(M_i) \subset N_i$ $\forall i$. Similarly we can talk about graded R-bimodules and their homomorphisms.

Example: BS_i is a graded bimodule for $R = \mathbb{C}[\mathfrak{h}^*]$ w. $\deg a \otimes b = i+j-1$ for $a \in R_i$, $b \in R_j$ (the shift is a convenient convention).

Constructions: • If B_1, B_2 are graded (bi)modules, then $B_1 \oplus B_2$ has a natural grading.

• Tensor product of two graded bimodules, $B \otimes_R B'$, is a graded bimodule w. $b_1(x \otimes y) b_2 := b_1 x \otimes y b_2$ & $\deg b \otimes b' = \deg b + \deg b'$. Similarly, for a graded R-module M, $B \otimes_R M$ is graded R-module: $b(x \otimes m) = bx \otimes m$. E.g. for $R = \mathbb{C}[\mathfrak{h}^*]$ we have the graded module $\underline{B} := B/B\mathfrak{h}$ $(= B \otimes_R \mathbb{C}_0)$. $f \in R$ acts via $f \mapsto f(0)$.

• Grading shift. For a graded R-(bi)module M and $j \in \mathbb{Z}$ we can define the graded (bi)module $M\langle j \rangle$ with same R-(bi)module structure but shifted grading: $M\langle j \rangle_i = M_{i+j}$. For example, $BS_i = \underbrace{R \otimes_{R^{s_i}} R}_{\text{w. its default grading}} \langle 1 \rangle$.

Let R-gr(bi)mod denote the category of finitely generated graded R-(bi)modules.

Exercise: (i) Show that for $B \in R$-grbimod, we have $\dim B_i < \infty$ $\forall i$.
(ii) Hom's in R-gr(bi)mod are finite dimensional (over $\mathbb{C}$).
(iii) Deduce that every object in R-gr(bi)mod decomposes into the

direct sum of indecomposables and any two decompositions have the same summands (up to isomorphism) – compare to Proposition from Sec 1.1 of Lecture 24.

(iv) If for $B \in R$-gr(bi)mod we have $B \simeq B\langle j\rangle$, then $j=0$.

1.3) Soergel (bi)modules: definition.

Set $R := \mathbb{C}[\mathfrak{h}^*]$. By a Soergel bimodule we mean a graded R-bimodule obtained from the BS_i's ($i=1,\ldots,n-1$) by using the following operations:

- taking $\oplus$
- taking $\otimes_R$
- applying grading shifts $\langle j\rangle$ ($j\in\mathbb{Z}$)
- taking direct summands in the category of graded R-bimodules.

Note that any Soergel bimodule is $\oplus$ of indecomposable Soergel bimodules in a unique way, see the previous exercise.

The category of Soergel bimodules is denoted by SBim.

By a Soergel module we mean a direct summand in $\underline{B}\,(= B/B\mathfrak{h})$ for $B\in$ SBim. Their category is denoted by SMod.

Remarks 1) In fact, one doesn't need to include direct summands in the definition of SMod: $B\in$ SBim is indecomposable $\Leftrightarrow \underline{B}\in$ SMod is. $\Leftarrow$ is very basic (exercise), $\Rightarrow$ is a result of Soergel. Moreover, $\underline{B}_1 \simeq \underline{B}_2 \Rightarrow B_1\simeq B_2$. We'll discuss these in the complement section.

2) For a word $\underline{w}=(s_{i_1},\dots,s_{i_k})$ - not necessarily reduced - define the Bott-Samelson bimodule $BS_{\underline{w}}:=BS_{i_k}\otimes_R BS_i \otimes_R \dots \otimes_R BS_{i_1}$. Then every $B\in SBim$ is $\bigoplus$ of indecomposable direct summands of $BS_{\underline{w}}$ w. grading shifts.

3) Let $B\in SBim$. We claim that $\forall f\in R^W$, the left action of f on B coincides with its right action. By 2) one needs to check this on $BS_{\underline{w}}$. There one reduces to the individual BS_i's, where it holds b/c $f\in R^W\subset R^{s_i}$. An R-bimodule is the same thing as an $R\otimes R$-module and the $R\otimes R$-action on B factors through

$$R\otimes R/(f\otimes 1-1\otimes f\mid f\in R^W)=R\otimes_{R^W}R.$$

Exercise: Observe that $R\otimes_{R^W}R$ is a finitely generated right R-module and deduce that every $B\in SBim$ is a finitely generated right R-module, and so every $M\in SMod$ is finite dimensional.

1.4) Connection to $\mathcal{O}^x$-proj.

The following claim is fairly basic and will be checked in the complement section.

Fact: Let M be a finite dimensional graded R-module. If it's indecomposable as a graded module, then it's indecomposable as an ordinary module (the opposite implication is manifest).

Let $SMod_{ungr}$ denote the category, where the objects are Soergel modules and morphisms are all R-linear homomorphisms (not necessarily graded). Soergel's theorems mean that $\mathbb{V}: \mathcal{O}^\chi\text{-proj} \xrightarrow{\sim} SMod_{ungr}$

Remark: One can formally recover Hom's in $SMod_{ungr}$ from those in $SMod$ and the grading shift functor as follows. Let $M,N \in R\text{-grmod}$. Then the vector space $Hom_R(M,N)$ is graded (as a vector space):
$Hom_R(M,N)_j = \{\varphi \in Hom_R(M,N) \mid \varphi(M_i) \subset N_{i+j}\} = Hom_{R\text{-grmod}}(M, N\langle j\rangle)$.
So $Hom_R(M,N) = \bigoplus_{j\in\mathbb{Z}} Hom_{R\text{-grmod}}(M, N\langle j\rangle)$.

The category $SBim$ also has a representation theoretic interpretation (where we need to use certain "Harish-Chandra bimodules" instead of objects in $\mathcal{O}^\chi$). This may be explained in a bonus lecture. What is important for us is that $SBim$ is the "Hecke category" - it "categorifies" the Hecke algebra $\mathcal{H}_v(S_n)$. This (as well as examples of indecomposables in $SBim$) will be explained in the next lecture

2) Complements

2.1) Proof of $P(w_0\cdot\lambda) \simeq \mathcal{T}_{-\rho\to\lambda}\Delta(-\rho)$.

We use Fact 3 mentioned in Sec 1.4 of Lec 23: $\mathcal{T}_{\lambda\to-\rho}(L(w\cdot\lambda)) = L(-\rho)$ if $w=w_0$ and zero else. So $Hom_{\mathcal{O}^\chi}(\mathcal{T}_{-\rho\to\lambda}\Delta(-\rho), L(w\cdot\lambda)) = [\text{adjunction}] = Hom_{\mathcal{O}^{-\rho}}(\Delta(-\rho), \mathcal{T}_{\lambda\to-\rho}L(w\cdot\lambda))$
$= [\text{every object in } \mathcal{O}^{-\rho} \text{ is } \Delta(-\rho)^{\oplus ?}, \text{Prob 1 in HW3}] = \mathbb{C}^{\oplus\delta_{w,w_0}}$. An isom'm $P(w_0\cdot\lambda) \simeq \mathcal{T}_{-\rho\to\lambda}\Delta(-\rho)$ follows.

2.2) Indecomposability for graded and ordinary modules.

Let M be a finite dimensional graded R-module. Assume it is indecomposable as a graded module. We claim it is indecomposable as an ordinary R-module.

Note that $End_R(M)$ is a graded $\mathbb{C}$-algebra w. grading introduced in Sec 1.4. We claim that $\operatorname{rad} End_R(M)$ is a homogeneous ideal. For this we use the following classical construction: to give a grading on a finite dimensional $\mathbb{C}$-vector space, V, is the same thing as to equip V with a rational $\mathbb{C}^\times$-action (on the graded component V_i the group $\mathbb{C}^\times$ acts by $t \mapsto t^i$). To give an algebra grading on a finite dimensional algebra (such as $End_R(M)$) is to give a rational representation of $\mathbb{C}^\times$ by algebra automorphisms (exercise). And, by its definition, the radical is stable under any automorphism. In particular, it's $\mathbb{C}^\times$-stable, hence graded.

Picking a direct summand in M amounts to picking an idempotent in $End_R(M)/\operatorname{rad} End_R(M)$ (that can be then lifted to $End_R(M)$. Picking a graded direct summand requires a degree 0 idempotent. So our claim becomes the following:

(*) Suppose that a semisimple $\mathbb{C}$-algebra A has an algebra grading s.t. $A_0 = \mathbb{C}1$. Then $A = \mathbb{C}$.

Here's the most elementary proof of (*): recall that the trace pairing $(x,y) \to tr_A(xy)$ is non-degenerate. Under this pairing, A_i and A_{-i} are dual to each other. Let $a \in A_i$, $b \in A_{-i}$ be s.t. $tr(ab) \neq 0$. Since $A_0 = \mathbb{C}1$, we must have that ab is a nonzero multiple of 1. But any element of A_i, $i \neq 0$, is nilpotent (b/c A is finite dimensional). So the only possibility is $A_i = 0$ for $i \neq 0$, and hence $A = A_0 = \mathbb{C}1$.

2.3) Indecomposables in SBim vs indecomposables in SMod.

Theorem 1 (Soergel): $B \mapsto \underline{B}$ defines a bijection between indecomposable Soergel bimodules (up to iso) and indecomposable Soergel modules.

To prove this theorem we need a lemma and another theorem.

Lemma: Any $B \in SBim$ is a free right R-module.

Proof: Observe R is a free rk 2 R^s-module for any $s = s_i$. It follows that $BS_{\underline{w}}$ is a free rk 2^k right R-module (where k is the length of $\underline{w}$). Any indecomposable object in SBim is a direct summand in some $BS_{\underline{w}}$ so is a graded projective (hence free) R-module.

Theorem 2: For B_1, B_2, the R-module $Hom_{R\text{-bimod}}(B_1, B_2)$ (from the right R-action on B_2) is free and

$$Hom_{R\text{-bimod}}(B_1, B_2) \otimes_R \mathbb{C}_0 \xrightarrow{\sim} Hom_{R\text{-mod}}(\underline{B}_1, \underline{B}_2).$$

Sketch of proof of Thm 1 mod Thm 2:

Thx to $End_{R\text{-bimod}}(B) \otimes_R \mathbb{C}_0 \simeq End_{R\text{-mod}}(\underline{B})$ we can lift a homogeneous idempotent in $End_{R\text{-mod}}(\underline{B})$ to a homogeneous idempotent in $\mathrm{End}_{R\text{-bimod}}(B)$. So if $\underline{B}$ is indecomposable, then so is B.

A proof of $\underline{B}_1 \simeq \underline{B}_2 \Rightarrow B_1 \simeq B_2$ is similar: we can lift a homogeneous isomorphism in $\mathrm{Hom}_{R\text{-mod}}(\underline{B}_1, \underline{B}_2)$ to a homogeneous isomorphism in $\mathrm{Hom}_{R\text{-bimod}}(B_1, B_2)$.

We don't prove Theorem 2. One proof requires deformations of $\mathbb{V}$ and $\mathcal{O}^\vee$. It's explained in my hand-written note for the $\mathcal{O}$-seminar (see ref. for Lec 24), Oct 24 meeting. A closely related result is Soergel's Hom formula, see [EMTW], Section 5.5.