Hecke algebra/category, Part IX.

1) Soergel (bi)modules, cont'd (but not quite finished...)

1.0) Recap: Set $R=\mathbb{C}[\mathfrak{h}^*]$. For a simple reflection $s=(i,i+1)\in W$ we consider the subalgebra of s-invariant polynomials $R^s\subset R$ and the graded Bott-Samelson bimodule $BS_s:=R\otimes_{R^s}R\langle 1\rangle$ (the grading shift explicitly means that $\deg 1\otimes 1=-1$). We have also defined more general Bott-Samelson bimodules: for $\underline{w}=(s_{i_1},\dots s_{i_k})$

$$BS_{\underline{w}}:=BS_{s_{i_k}}\otimes_R BS_{s_{i_2}}\otimes_R\dots\otimes_R BS_{i_1}$$

A Soergel bimodule is a direct sum of shifts of direct summands of various $BS_{\underline{w}}$'s. The full subcategory of Soergel bimodules in R-grbimod is denoted by SBim. By the construction, SBim is closed under $\cdot\otimes_R\cdot$, $\oplus$, $\langle ?\rangle$ and also taking direct summands. Every Soergel bimodule is the direct sum of indecomposables in a unique way.

Questions we plan to address in this lecture:

Q: How are the indecomposables classified?

A: Up to a grading shift, the indecomposables are classified by the elements of W. For $w\in W$, let B_w be the corresponding indecomposable Soergel bimodule.

Q: How to decompose $B_u\otimes B_w$ into $\oplus$ indecomposables.

A: "The same way" as to decompose $C_u C_w \in \mathcal{H}_v(W)$ into the $\mathbb{Z}[v^{\pm 1}]$-linear combination of C_y's. Here C_w's are the Kazhdan-Lusztig basis elements.

This is one of the ways how SBim is a "Hecke category."

1.1) Decompositions of some $BS_{\underline{w}}$'s.

Lemma 1: Let s be a simple reflection in W. Then
$$BS_s \otimes_R BS_s \simeq BS_s\langle -1\rangle \oplus BS_s\langle 1\rangle.$$

Proof: Let $s=(i,i+1)$, $h=h_i$. Then, as an R^s-bimodule, $R = R^s \oplus hR^s$ (exercise) hence R^s w. generator in deg 2.
$$R \simeq R^s \oplus R^s\langle -2\rangle \qquad (1)$$
So,
$$BS_s \otimes_R BS_s\langle -2\rangle = R\otimes_{R^s} R \cancel{\otimes_R R} \otimes_{R^s} R = R \otimes_{R^s} R \otimes_{R^s} R = [(1)]$$
$$= R\otimes_{\cancel{R^s}} \cancel{R^s} \otimes_{R^s} R \oplus R\otimes_{\cancel{R^s}} \cancel{R^s}\langle -2\rangle \otimes_{R^s} R = BS_s\langle -1\rangle \oplus BS_s\langle -3\rangle.$$
This implies the claim of the lemma (apply $\langle 2\rangle$). $\square$

Remark 1: i) BS_s is an indecomposable graded bimodule. Indeed, by the easy part of Remark 1 in Sec 1.3 of Lec 25, BS_s is indecomposable as long as $\underline{BS}_s = BS_s \otimes_R \mathbb{C}_0 = R\otimes_{R^s} \mathbb{C}_0$ is an indecomposable R-module. The module $\underline{BS}_s$ is 2-dimensional w. basis $1\otimes 1$ & $h\otimes 1$. Note that $1\otimes 1$ generates $\underline{BS}_s$ as an R-module: $h(1\otimes 1) = h\otimes 1$. If $\underline{BS}_s$ decomposes as $M^1 \oplus M^2$ for graded R-modules, then $(\underline{BS}_s)_{-1} \simeq M^1_{-1} \oplus M^2_{-1}$. But $(\underline{BS}_s)_{-1} = \mathbb{C}\, 1\otimes 1$ so $1\otimes 1$ is forced to be in one of M^1 & M^2, say M^1. Since $1\otimes 1$ is a generator, we then have $\underline{BS}_s = M^1$.

ii) Note that for $C_s = H_s + v \in \mathcal{H}_v(W)$ have $C_s^2 = (v+v^{-1})C_s$.

Lemma 2: Let $s\neq t$ be simple reflections. Then $BS_{(s,t)}$ is indecomposable.

Proof: Step 1: For $B\in R\text{-grbimod}$ let's understand the bimodule structure on $BS_s\otimes_R B = R\otimes_{R^s} B\langle 1\rangle$ in terms that of B.

$R = R^s \oplus R^s h_s$ as R^s-bimodule w. basis $1, h_s$ ($=h_i$ if $s=s_i$). So $R\otimes_{R^s} B\langle 1\rangle$ is spanned, as a left R^s-module, by elements $1_s\otimes b := 1\otimes b$ ($\deg = \deg b - 1$) & $h_s\otimes b$ ($\deg = \deg b+1$), where b is a homogeneous element of B. The product by el'ts of $\mathfrak{h}\subset \mathbb{C}[\mathfrak{h}^*]$ is as follows. We can decompose $\mathfrak{h}$ as $\mathbb{C}h_s \oplus \mathfrak{h}^s$. For $r\in R^s$ (in particular, $r\in \mathfrak{h}^s$) we have

$$(2)\quad \begin{cases} r(?\otimes b) = ?r\otimes b = ?\otimes rb \quad (? = 1 \text{ or } h_s) \\ h_s(1_s\otimes b) = h_s\otimes b,\ \ h_s(h_s\otimes b) = h_s^2\otimes b = [h_s^2\in R^s] = 1_s\otimes h_s^2 b. \end{cases}$$

Step 2: In particular, for $\underline{w} = (s_{i_1},\dots,s_{i_k})$, $BS_{\underline{w}}$ is a free right R-module w. basis $?_k\otimes ?_{k-1}\otimes\dots\otimes ?_1$ w. $?_i\in\{1_{s_i}, h_{s_i}\}$. The degree is #h's − #1's. The left R-module structure can be recovered from (2). The tensors above form a vector space basis in $\underline{BS}_{\underline{w}} = BS_{\underline{w}}/BS_{\underline{w}}\mathfrak{h}$.

Step 3: Apply this to $w=(s,t)$. We need to show that $\underline{BS}_{(s,t)}$ is an indecomposable graded R-module. We have two cases: $st=ts$ (easier, left as an exercise) and $st\neq ts$. The latter reduces to $n=3$ (exercise).

Let $s=s_1$, $t=s_2$. We have $h_1=(1,-1,0)$, $h_2=(0,1,-1)$. Set $y_1=(1,1,-2)$ $y_2=(2,-1,-1)$, so that $s_i y_i = y_i$, and $\{h_i, y_i\}$, $i=1,2$, are bases of $\mathfrak{h}$. We claim that $\underline{BS}_{(s,t)}$ is generated by $1\otimes 1$, this implies that it's indecomposable (compare to i) of Remark 1).

- $h_2(1_2\otimes 1_1) = h_2 \otimes 1_1$
- $y_2(1_2\otimes 1_1) = 1_2\otimes y_2 1_1 = [y_2 = \alpha h_1 + \beta y_1$ w. $\alpha \neq 0$; $1_2\otimes y_1 = 0$ b/c $y_1 \in R^{s_1}$ & has positive degree$] = \alpha \cdot 1_2 \otimes h_1 \Rightarrow$
- $h_2 y_2 (1_2 \otimes 1_1) = \alpha h_2 \otimes h_1$ $\square$

Remark 2: We've seen that for $n=3$, we have $C_t C_s = C_{ts}$, example in Sec 1.3 of Lec 21.

Lemma 3: Use the notation of the previous lemma and assume $st \neq ts$. We have $BS_{(s,t,s)} \simeq R\otimes_{R^{s,t}} R\langle 3\rangle \oplus BS_t$ (here $R^{s,t} := R^s \cap R^t$, i.e the subalgebra of invariants for $\langle s,t\rangle (\simeq S_3) \subset W$). Moreover, $R\otimes_{R^{s,t}} R$ is indecomposable.

Instead of a proof: Note that $\underline{BS}_{(s,t,s)}$ is an 8-dimensional R^{coW}-module. We have $R^{coW} 1_s \otimes 1_t \otimes 1_s \simeq R^{coW}$. One can split it as a direct summand, the complement is 2-dimensional and it's $R\otimes_{R^t} \mathbb{C}_0$. With yet more work one lifts this to an isomorphism as in the lemma. For a proof see [EMTW], Example 4.41.

Remark 3: $C_s C_{ts} = C_{sts} + C_t$, Example in Sec 1.3 of Lec 21.

1.2) Indecomposable Soergel bimodules

The following result is from the original paper of Soergel.

Theorem: Up to grading shift, the indecomposable Soergel bimodules are classified by the elements of W. More precisely, $\forall\, w\in W\ \exists!$ indecomposable $B_w \in SBim$ s.t. $\forall$ reduced expression $\underline{w}$ of w we have $BS_{\underline{w}} = B_w \oplus \bigoplus_{u<w} B_u\langle ?\rangle^{\oplus ?}$

Remark: Compare to the decomposition of $\Theta_{\underline{w}}\Delta(\lambda)$, Sec 1.3 of Lec 24.

Example: $W=S_3$, $s=(1,2)$, $t=(2,3)$. Then $B_1=R$, $B_s=BS_s$, $B_t=BS_t$, $B_{st}=BS_{(s,t)}$, $B_{ts}=BS_{(t,s)}$, $B_{w_0} = R\otimes_{R^W} R\langle 3\rangle$. There are no other indecomposables. This will follow if we show that for all $w\in W$, $i=1,2$, $BS_i\otimes_R B_w$ has no new indecomposable summands. This is an easy check: e.g for $w=w_0$, we have $B_{w_0}\subseteq_{\oplus} BS_{(s,t,s)} \Rightarrow$
$BS_s\otimes_R B_{w_0}\subseteq_{\oplus} BS_{(s,s,t,s)} = [BS_{(s,s)} = BS_s\langle -1\rangle\oplus BS_s\langle 1\rangle] = BS_{(s,t,s)}\langle -1\rangle\oplus BS_{(s,t,s)}\langle 1\rangle$.

1.3) Split K_0 & Soergel's categorification theorem.

Define the split Grothendieck group $K_0(SBim)$ as the group generated by symbols $[B]$ for $B\in SBim$ (up to iso) and relations $[B_1\oplus B_2] = [B_1]\oplus[B_2]$. Since every $B\in SBim$ is uniquely presented as $\oplus$ of indecomposables, $K_0(SBim)$ is a free abelian group whose basis is the classes of indecomposables. This group has the following structures.

- The unique $\mathbb{Z}[v^{\pm 1}]$-module structure w. $v[B]=[B\langle -1\rangle]$.
- The unique associative ring structure $[B_1][B_2]=[B_1\otimes_R B_2]$

Together they give a $\mathbb{Z}[v^{\pm 1}]$-algebra structure on $K_0(SBim)$. What algebra do we get?

Theorem (Soergel): 1) There's a unique $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphism $\mathcal{H}_v(W)\longrightarrow K_0(SBim)$ w. $C_s=H_s+v\mapsto [BS_s]$ $\forall$ simple reflection $s\in W$. It's an isomorphism.

2) Under this isomorphism, $C_w\mapsto [B_w]$, $\forall$ $w\in W$.

Sketch of proof of 1): The algebra $\mathcal{H}_v(W)$ is generated by the elements $C_{s_i}=H_{s_i}+v$ subject to the following relations:

(i) $C_{s_i}^2=(v+v^{-1})C_{s_i}$

(ii) $C_{s_i}C_{s_j}=C_{s_j}C_{s_i}$ if $|i-j|>1$

(iii) $C_{s_i}C_{s_j}C_{s_i}-C_{s_j}=C_{s_j}C_{s_i}C_{s_j}-C_{s_i}$ if $|i-j|=1$.

These relations hold for $[BS_{s_i}]$: (i) follows from Lemma 1, (ii) from the easy case of Lemma 2. Finally, both sides of (iii) in $K_0(SBim)$ coincide w. $[B_w]$ for $w=s_js_is_j=s_is_js_i$.

$\mathcal{H}_v(W)\twoheadrightarrow K_0(SBim)$ b/c the span of the classes $[BS_{\underline{w}}]$ contains all $[B_w]$ by Theorem in Sec 1.2. By that theorem, $[B_w]$'s form a basis of $K_0(SBim)$ as a $\mathbb{Z}[v^{\pm 1}]$-module. Also $\mathcal{H}_v(W)\cong_{\mathbb{Z}[v^{\pm 1}]}\mathbb{Z}[v^{\pm 1}]^{\oplus |W|}\cong_{\mathbb{Z}[v^{\pm 1}]}K_0(SBim)$. Any surjective $\mathbb{Z}[v^{\pm 1}]$-linear endomorphism of a finitely generated $\mathbb{Z}[v^{\pm 1}]$-module is an isomorphism. So

$$\mathcal{H}_v(W)\xrightarrow{\sim} K_0(SBim). \qquad \square$$

1.4) Remarks

I) 1) of the theorem justifies the name "Hecke category" for SBim. It's relatively basic. In a later paper: W. Soergel

Soergel proved this using essentially Commutative algebra (but the proof is informed by the representation theoretic/geometric setting).

II) SBim is the first example of an "algebraic categorification." Another important example discovered later: the 2-Kac-Moody algebras (Khovanov-Lauda, Rouquier). They're related to the quantum groups in the same fashion as SBim is related to $\mathcal{H}_v(W)$.

III) 2) is a much deeper result. Soergel's original proof relied on an interpretation of SBim in terms of perverse sheaves on the flag variety and using tools from perverse sheaves such as the BBDG (Beilinson-Bernstein-Deligne-Gabber) decomposition theorem. In their 2014 Annals paper Elias and Williamson managed to provide a fully algebraic proof emulating the geometry.

IV) The algebra $\mathcal{H}_v(W)$ on its own doesn't "know" about any basis, meaning we cannot recover any basis just from the algebra structure. But once an isomorphism $\mathcal{H}_v(W) \xrightarrow{\sim} K_0(\text{SBim})$ is established, $\mathcal{H}_v(W)$ acquires a basis: the classes of indecomposables in SBim (w. some shifts). Those are naturally recovered from the category structure!

V) Part 2) & results from the previous two lectures imply the KL conjecture. Namely, we can form the split K_0 of SMod and of SMod_{ungr}. The former is $\mathcal{H}_v(W)$, viewed as a regular right module over $\mathcal{H}_v(W) = K_0(\mathrm{SBim})$, where SBim acts on SMod by tensoring over R.
$K_0(\mathrm{SMod}_{ungr}) \xrightarrow{\sim} K_0(\mathrm{SMod}) / \underbrace{(v-1)}_{\text{the grading shift became trivial}} K_0(\mathrm{SMod}) = \mathbb{Z}W$.

Since $\mathrm{SMod}_{ungr} \xleftarrow{\sim} \mathcal{O}^{\chi}_{-}\text{proj}$, we get $K_0(\mathcal{O}^{\chi}_{-}\text{proj}) = \mathbb{Z}W$ w. right action of W recovered from $[P](s_i+1) = [\Theta_i P]$.

While $\Delta(w\cdot\lambda) \notin \mathcal{O}^{\chi}_{-}\text{proj}$ for $w \neq 1$, we still have a well-defined class $[\Delta(w\cdot\lambda)] \in K_0(\mathcal{O}^{\chi}_{-}\text{proj})$ thx to the Verma filtrations on $P(w\cdot\lambda)$'s & their upper triangularity properties, Sec 1.2 of Lec 24. In particular $[P(u\cdot\lambda)] = \sum_{w\in W} (\text{mult of } \Delta(w\cdot\lambda) \text{ in } P(u\cdot\lambda))[\Delta(w\cdot\lambda)]$. By the SES in the end of the proof of Proposition in Sec 1.3 of Lec 23,

$$[\Theta_i \Delta(w\cdot\lambda)] = [\Delta(ws_i\cdot\lambda)] + [\Delta(w\cdot\lambda)].$$

It follows that under the identification $K_0(\mathcal{O}^{\chi}_{-}\text{proj}) \simeq \mathbb{Z}W$, $[\Delta(w\cdot\lambda)]$ goes to w. So the multiplicity of $\Delta(w\cdot\lambda)$ in $P(u\cdot\lambda)$ is the coefficient of H_w in C_u (which is the KL polynomial $c_{u,w}(v)$) evaluated at $v=1$. Thx to Sec 1.2 in Lec 24, this claim is equivalent to the KL conjecture.

VI) We have been dealing with Soergel bimodules over $\mathbb{C}$. We could have chosen any char 0 field and Theorem above still holds. Or we can try a characteristic p field. For p not too small, say, $p>n$ certainly works, SBim as defined above is still a reasonable object. Theorem in Sec 1.2 as well as 1) of Theorem in this section continue to hold.

2) of the theorem – which is the main part – fails however (a geometric reason: the decomposition theorem fails in char p).

Instead the indecomposables in SBim give a new basis in $H_v(W)$, called the p-KL basis. Unlike the usual KL basis, there are no known combinatorial formulas for the p-KL basis and finding ones is a major open problem in the field.

VII*) Why should we care about p-KL polynomials? Well, for $W=S_n$ they aren't that useful. This is because in modular representation theory interesting categories such as representations of simple algebraic groups and their Lie algebras are controlled by affine Weyl groups (for SL_n & $\mathfrak{sl}_n$ this is $W(\tilde{A}_n)$). One can define SBim for any Coxeter group but the original Soergel's definition is useless if char $\mathbb{F}>0$ and the Weyl group is affine (a reason: the natural representation is not faithful, the translation by every lattice element divisible by p acts by 0). The category SBim needs a modification. The resulting category, first constructed by Elias and Williamson has several equivalent definitions. Two of them are algebraic:

B. Elias, G. Williamson "Soergel calculus" Repres. Theory (2016) – the original definition, where SBim is presented by generators & relations. Lots of fun pictures – similar to what knot theorists use.

N. Abe "A bimodule description of the Hecke category" Compos. Math

157 (2021) is the most recent and also most elementary equivalent definition: to fix the pathologies w. Soergel's original definition, Abe adds just a bit of extra data that is immediately available in char 0.

Now let's get to the question in the beginning of this part: why to care about p-KL basis? The answer: for the same reason that we care usual ones. They (or rather their values at ±1) give multiplicities in various categories of interest for Rep. theory. The case of rational representations of reductive algebraic groups has been studied most extensively (this is the last five years or so).