Hecke algebra/category, Part X.

1) Varieties
2) Cohomology vs Soergel modules.

1.0) Introduction

In this lecture we'll investigate the geometry behind the Soergel theory. The starting object in this - as well as in most of the geometric developments of Lie representation theory — is the flag variety.

Definition: As a set, the flag variety $\mathcal{F}\ell_n(\mathbb{C})$ consists of complete flags of subspaces $\{0\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = \mathbb{C}^n$ w. $\dim V_i = i$. This is a projective algebraic variety (so a compact topological space).

Remark: We have already encountered $\mathcal{F}\ell_n$ but over $\mathbb{F}_q$. This was in Lecture 18, when we've first encounted the Hecke algebra. Namely note that, for any field $\mathbb{F}$, the $G = GL_n(\mathbb{F})$-action on $\mathbb{F}^n$ gives an action on $\mathcal{F}\ell_n(\mathbb{F})$. It's transitive (exercise). For the standard flag - $V_i = \mathrm{Span}_{\mathbb{F}}(e_1, \ldots, e_i)$ - where $e_1, \ldots, e_n$ is the tautological basis of $\mathbb{F}_n$, the stabilizer of this flag in G is the Borel subgroup B. So, as a set $\mathcal{F}\ell_n(\mathbb{F})$ is identified w. G/B. In fact, if $\mathbb{F}$ is algebraically closed, then G/B has a natural algebraic variety structure and $G/B \xrightarrow{\sim} \mathcal{F}\ell_n(\mathbb{F})$ is a variety isomorphism.

1.1) Schubert varieties.

The base field is $\mathbb{C}$.

Recall that $G/B \simeq \coprod_{w\in W} BwB/B$, where BwB/B is the Schubert cell identified w. $\mathbb{C}^{\ell(w)}$ (see Sec 3.1 of Lec 18).

Definition: The Schubert variety (associated to w) is $\overline{BwB/B} \subset G/B$, where the closure is taken in Zariski topology (the closure in the usual topology gives the same).

Since $G/B = \mathcal{F}\ell_n(\mathbb{C})$ is a projective variety, so is $\overline{BwB/B}$.

Let's describe BwB/B and $\overline{BwB/B}$ in linear algebraic terms. Let $\mathcal{F} = (\{0\}=V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n)$, $\mathcal{F}' = (\{0\}=V_0' \subset V_1' \subset \ldots \subset V_n' = \mathbb{C}^n)$ be two flags. We can read a permutation from them as follows. For $w \in W (=S_n)$, we say $\mathcal{F}, \mathcal{F}'$ are in relative position w if

$$\dim(V_i \cap V_j') = \#(\{1,\ldots,i\} \cap w\{1,\ldots,j\}),\ \forall i,j.$$

Exercise: 1) Prove that $\forall\ \mathcal{F}, \mathcal{F}'$ such w exists and is unique.

2) Show that TFAE:

- $\mathcal{F}, \mathcal{F}'$ are in relative position w.
- $\exists$ basis $v_1,\ldots,v_n$ of $\mathbb{C}^n$ s.t. $V_i = \mathrm{Span}_{\mathbb{C}}(v_1,\ldots,v_i)$ & $V_j' = \mathrm{Span}(v_{w(1)},\ldots,v_{w(j)})$

Let $\mathcal{F}^{st}$ denote the standard flag (w. $V_i = \mathrm{Span}_{\mathbb{C}}(e_1,\ldots,e_i)$). We have the following result

Lemma: 1) BwB/B consists of all flags that in relative position w with $\mathcal{F}^{st}$.

2) $\overline{BwB}/B = \coprod_{u \leq w} BuB/B =$
$\{\mathcal{F} = (V_j) \in \mathcal{F}\ell_n \mid \dim(V_i^{st} \cap V_j) \geq \#(\{1,\dots,i\} \cap w\{1,\dots,j\})\}$

Sketch of proof: 1) Note that $M_w B/B$ is the flag given by $V_j = \mathrm{Span}(e_{w(1)},\dots,e_{w(j)})$. Set $\mathcal{F}\ell(w) := \{\mathcal{F}$ of relative position w w. $\mathcal{F}^{st}\}$. The locus $\mathcal{F}\ell(w)$ is B-stable b/c $\mathcal{F}^{st}$ is fixed by B. It follows that this locus contains BwB/B and is a union of some Schubert cells. There are $|W|$ of such loci and $|W|$ Schubert cells. So $\mathcal{F}\ell(w) = BwB$.

2) -more sketchy: one shows that the last two subsets are the same, which amounts to a combinatorial description of the Bruhat order. Then one shows that the first term is contained in the third (the dimensions of intersections can only grow when we "degenerate" the flag). The second is contained in the first. Indeed, let $i<j$ be such that $(ij)w < w \Leftrightarrow w^{-1}(i) > w^{-1}(j)$. Consider the map $\gamma: \mathbb{C} \longrightarrow GL_n(\mathbb{C}): x \mapsto I + xE_{w^{-1}(i),w^{-1}(j)}$. Note that $\gamma(t)wB/B \subset BwB$ ($\gamma(t) \subset U_w$ from Sec 3.1 of Lec 18). Then one can show that $\overline{\gamma(t)wB/B} \cap B(ij)wB/B \neq \emptyset$ forcing $B(ij)wB/B \subset \overline{BwB}$ $\square$

Example: 1) Bw_0B/B is open in $G/B = \mathcal{F}\ell_n$, irreducible variety. So $\overline{Bw_0B/B} = G/B$.

2) Let $n=3$, $s=(1,2)$, $t=(2,3)$. Then $\overline{BwB/B}$ consists of all $\mathcal{F}=\{0=V_0\subset V_1\subset V_2\subset V_3=\mathbb{C}^3\}$ s.t.

- $w=1$: $V_i=V_i^{st}$ for $i=1,2$. A point.
- $w=s$: $V_2=V_2^{st}$; $\simeq\mathbb{P}^1$.
- $w=t$: $V_1=V_1^{st}$; $\simeq\mathbb{P}^1$.
- $w=st=(3,1,2)$: $V_1\subset V_2^{st}$; will describe later.
- $w=ts=(1,3,2)$: $V_1^{st}\subset V_2$; — · — · — · — · — · —

1.2) Bott-Samelson varieties

One issue w. Schubert varieties is that they are singular (i.e. not manifolds). The issue first arises when $n=4$, e.g. $\overline{BwB/B}$ is singular for $w=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$. For a singular variety X one usually tries to find resolution of singularieties: a smooth variety $\tilde{X}$ w. a morphism $\pi:\tilde{X}\to X$ s.t.

- π is proper (the preimage of every compact subset is compact)
- π is birational (it's an isomorphism over a Zariski open dense subset).

We will define Bott-Samelson varieties, $BS_{\underline{w}}$, where $\underline{w}=(s_{i_1},\dots,s_{i_k})$ is a word in simple reflections. When $\underline{w}$ is a reduced expression of w, we'll see that $BS_{\underline{w}}$ is a resolution of singularities of $\overline{BwB/B}$.

Definition: As a set $BS_{\underline{w}}$ consists of $(k+1)$-tuples $(\mathcal{F}^0,\dots,\mathcal{F}^k)$, $\mathcal{F}^i=(0=V_0^i\subset V_1^i\subset\dots\subset V_k^i\subset\dots\subset V_n^i=\mathbb{C}^n)$ with the following properties:

- $\mathcal{F}^0=\mathcal{F}^{st}$
- for each $\ell=1,\dots k$, we have $V_j^\ell=V_j^{\ell-1}$ for $j\neq i_\ell$.

Before we produce examples (for $n=3$), let's notice that $BS_{\underline{w}}$ admits two natural forgetful maps:

$\eta: BS_{\underline{w}} \to BS_{\underline{w}'}$, $\underline{w}' = (s_{i_1}, \dots, s_{i_{k-1}})$: forget the last flag

$\pi: BS_{\underline{w}} \to \mathcal{F}l_n$: forget all flags but the last one.

Exercise: η is a "$\mathbb{P}^1$-bundle" meaning, essentially, that every fiber of η is $\mathbb{P}^1$ (hint: to recover the last flag boils down to giving a 1-dim'l subspace inside a fixed 2-dim'l space).

This bundle is locally trivial in a suitable sense implying that $BS_{\underline{w}}$ is smooth. We also see that $BS_{\underline{w}}$ is a projective variety of dimension k. In particular, π is proper.

Example: $k=1$: $BS_s = \overline{BsB/B} \simeq \mathbb{P}^1$.

- $n=3$: $w = st$: we claim that $\pi: BS_{(t,s)} \xrightarrow{\sim} \overline{BtsB/B}$

We have $BS_{(t,s)} = (\mathcal{F}^{st}, (0 \subset V_1^{st} \subset V_2 \subset \mathbb{C}^3), (0 \subset V_1 \subset V_2 \subset \mathbb{C}^3))$. The flag $(0 \subset V_1 \subset V_2 \subset \mathbb{C}^3)$ satisfies the condition that $V_1^{st} \subset V_2$ characterizing $\overline{BtsB/B}$. Conversely given $(0 \subset V_1 \subset V_2 \subset \mathbb{C}^3)$ w. $V_1^{st} \subset V_2$ we recover an element of $BS_{(t,s)}$ uniquely.

- Similarly, $BS_{(s,t)} \xrightarrow{\sim} \overline{BstB/B}$. So both $\overline{BtsB/B}$, $\overline{BstB/B}$ are smooth.
- Now consider $\pi: BS_{(t,s,t)} \to G/B$. The variety $BS_{(t,s,t)}$ consists of triples (omit $\mathcal{F}^{st}$, also omit 0 and the full space:

$$(V_1^{st} \subset V_2'; V_1 \subset V_2'; V_1 \subset V_2)$$

We send this triple to $V_1 \subset V_2$. Let's determine the preimages under π.

There are 2 cases:

i) $V_1 \neq V_1^{st}$. Then we uniquely recover V_2' as $V_1 \oplus V_1^{st}$. So the preimage is a single point. In fact, π is an isomorphism over this locus in $\mathcal{Fl}_n$, which is exactly $\mathcal{Fl}_n \setminus \overline{BtB/B}$.

ii) $V_1 = V_1^{st}$. Then there's $\mathbb{P}^1$ choices from V_2' so the fiber is $\mathbb{P}^1$.

Theorem: Let $\underline{w} = (s_{i_1}, \ldots s_{i_\ell})$ be a reduced expression of w. Then the image of $\pi: BS_{\underline{w}} \longrightarrow \mathcal{Fl}_n$ is $\overline{BwB/B}$ and over BwB, π is a bijection (hence an isomorphism).

Sketch of proof: By induction on ℓ we reduce to proving the following: consider the set $\{\mathcal{F}', \mathcal{F}''\}$ s.t $\mathcal{F}'$ & $\mathcal{F}^{st}$ are in relative position $w' = s_{i_1} \ldots s_{i_{\ell-1}}$ & $\mathcal{F}''$ & $\mathcal{F}'$ are in relative position s_{i_ℓ}. Then $\mathcal{F}''$ & $\mathcal{F}^{st}$ are in relative position w & moreover $\mathcal{F}'$ is uniquely recovered from $\mathcal{F}''$. We want prove this but we've seen a similar fact before: when in Sec 3.2 of Lec 18 we've proved that $T_u T_w = T_{uw}$ provided $\ell(uw) = \ell(u) + \ell(w)$. $\square$

2) Cohomology vs Soergel modules.

2.1) Basics on cohomology

Let X be a topological space. To X one can assign an invariant, a $\mathbb{C}$-algebra $H^*(X) = \bigoplus_{i \geq 0} H^i(X)$, the cohomology, which is "graded-commutative" in the following sense:

Definition: Let $A = \bigoplus_{i \geq 0} A_i$ be a graded associative algebra. We say A is graded-commutative if $\forall\, a \in A_i, b \in A_j$ we have $ab = (-1)^{ij} ba$. In particular, if $A_i = \{0\}$ for i odd, then "graded commutative" is the same as commutative.

In fact, H^* is a contravariant functor from the category of topological spaces (even better, from the homotopy category) to the category of graded-commutative algebras. In particular, if $f: X \to Y$ is a continuous map, then we have a graded algebra homomorphism $f^*: H^*(Y) \longrightarrow H^*(X)$ (that only depends on f up to homotopy).

Example: Let $X = \mathcal{F}\ell_n(\mathbb{C})$. It's paved by affine spaces (Schubert cells) labelled by the elements of W. A general result implies that $\dim H^*(X) = |W|$. A more careful analysis shows that as an algebra $H^*(X)$ is nothing else but $\mathbb{C}[\mathfrak{h}^*]^{coW}$ w. $\mathfrak{h}$ in deg 2 (which is one explanation of why we choose the doubled grading on R in Lec 25). The images of the variables x_i in $\mathbb{C}[\mathfrak{h}^*]^{coW}$ are the 1st Chern classes of tautological line bundles on $\mathcal{F}\ell_n(\mathbb{C})$.

2.2) Cohomology of Bott-Samelson & Schubert varieties.

Let $\underline{w} = (s_{i_1}, \dots, s_{i_k})$. Note that $\pi: BS_{\underline{w}} \longrightarrow \mathcal{F}\ell_n(\mathbb{C})$ gives an algebra homomorphism $\pi^*: H^*(\mathcal{F}\ell_n(\mathbb{C})) \longrightarrow H^*(BS_{\underline{w}})$.

In particular, $H^*(BS_{\underline{w}})$ becomes a graded $H^*(\mathcal{F}\ell_n(\mathbb{C}))$-module.

Fact: As a graded $H^*(\mathcal{Fl}_n(\mathbb{C}))=\mathbb{C}[\mathfrak{h}^*]^{coW}$-module, $H^*(BS_{\underline{w}})\xrightarrow{\sim}\underline{BS}_{\underline{w}}\langle -k\rangle$ (note that the shift of grading in the definition of $\underline{BS}_{\underline{w}}$ is the "perverse shift"; note also that the isomorphism above is that of algebras).

One could then expect that the indecomposable Soergel module $\underline{B}_{w^{-1}}$ is $H^*(\overline{BwB/B})\langle \ell\rangle$ w. $\ell=\ell(w)$. This is the case when $\overline{BwB/B}$ is smooth. Otherwise $H^*(\overline{BwB/B})$ is not a correct object.

There is a number of properties that the cohomology of smooth projective varieties satisfy (MATH 618 in S23 will discuss this). The most basic one is the "Poincare duality": if M is a compact n-dimensional real manifold, which is orientable (complex manifolds automatically satisfy this property), then $H^i(M)\xrightarrow{\sim}H^{n-i}(M)^*$ (if you think about the cohomology in terms of differential forms, then multiply the forms & integrate). In particular, the dimensions of the graded pieces are symmetric about $n/2$. This doesn't need to be the case for $H^*(\overline{BwB/B})$ we have

$$\dim H^k(\overline{BwB/B})=\#\{u\in W\mid u\preceq w\ \&\ \ell(u)=k\}$$

b/c $\overline{BwB/B}$ is paved by affine spaces $BuB/B\simeq\mathbb{C}^{\ell(u)}$ for $u\preceq w$, Lemma in Sec 1.1. E.g. for $w=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$ of length 4, there are 4 permutations u of length 3 w. $u\preceq w$ and 3 permutations of length 1, so there's no symmetry.

One replaces $H^*(\overline{BwB/B})$ with the "intersection cohomology" $IH^*(\overline{BwB/B})$ (studied in the perverse sheaf class). It looses some basis properties (homotopy invariance, being an algebra), retains some others – still an $H^*(\overline{BwB/B})$-, hence $H^*(\mathcal{Fl}_n)$-module, and acquires additional properties such as Poincare duality.

Thm (Soergel): We have $IH^*(\overline{BwB/B}) \simeq \underline{B}_w$.

Remarks*: 1) One can incorporate the Soergel bimodules into this geometric picture: instead of the usual cohomology one considers the T-equivariant cohomology for the maximal torus $T \subset SL_n(\mathbb{C})$ & its natural action on the varieties of interest.

2) The decomposition $\underline{BS}_{\underline{w}} = \underline{B}_w \oplus \bigoplus_{u<w} \underline{B}_u\langle ?\rangle^{\oplus ?}$ comes from the BBDG decomposition theorem applied to $\mathcal{BS}_{\underline{w}} \longrightarrow \overline{Bw^{-1}B/B}$.

3) The BBDG theorem is stated for the so called "perverse sheaves" not just their cohomology. A deep result of Soergel is that the hypercohomology functor $\mathbb{H}^*$ gives rise to an equivalence between:

- The full subcategory $\mathcal{D}^b_c(\mathcal{Fl}_n(\mathbb{C}))$ whose objects are $\bigoplus IC(BwB/B)[?]^{\oplus ?}$
- SMod.