Representations of symmetric groups, part 3.

0) Recap
1) Uniqueness of weights.
2) Varying the path.
3) Degenerate affine Hecke algebra.

0) $Z_m(n) := \{z \in \mathbb{C}S_n \mid az = za \ \forall a \in \mathbb{C}S_m\}$

$\forall V \in \operatorname{Irr}(\mathbb{C}S_n), U \in \operatorname{Irr}(\mathbb{C}S_m)$, the space $\operatorname{Hom}_{\mathbb{C}S_m}(U,V)$ is an irreducible $Z_m(n)$-module w. action given by
(1) $[z\varphi](u) := z[\varphi(u)] \ \forall z \in Z_m(n), \varphi \in \operatorname{Hom}_{\mathbb{C}S_m}(U,V), u \in U$.

Theorem: The algebra $Z_m(n)$ is generated by:
- $Z_m(m)$, a subalgebra in the center
- $\mathbb{C}S_{[m+1,n]}$
- The Jucys-Murphy elements $J_k := \sum_{i=1}^{k-1}(i,k)$ for $m+1 \leqslant k \leqslant n$.

Corollary 1: 1) $Z_{n-1}(n)$ is commutative.
2) $\forall U \in \operatorname{Irr}(\mathbb{C}S_{n-1}), V \in \operatorname{Irr}(\mathbb{C}S_n)$, the multiplicity of U in V is 0 or 1.
3) If U occurs in V, then J_n acts on U by scalar.

We've defined the branching graph & can talk about paths

$V^m \in \mathrm{Irr}(\mathbb{C}S_m)$, $V^n \in \mathrm{Irr}(\mathbb{C}S_n) \rightsquigarrow \mathrm{Path}(V^m, V^n) = \{V^m \to V^{m+1} \to \dots \to V^n\}$

$\bar{P} \in \mathrm{Path}(V^m, V^n) \rightsquigarrow \mathbb{C}S_m$-submodule $V^m(\bar{P}) \subset V^n$,

$$V^n = \bigoplus_{V^m} \bigoplus_{\bar{P} \in \mathrm{Path}(V^m, V^n)} V^m(\bar{P})$$

$\varphi_{\bar{P}} :=$ embedding $V^m(\bar{P}) \hookrightarrow V^n$.

Weight $w_{\bar{P}} = (w_{m+1}, \dots w_n)$: $\bar{P} = (V^m \to V^{m+1} \to \dots \to V^n)$

$w_i :=$ scalar by which J_i acts on V^{i-1} inside V^i.

Lemma: The elements $\varphi_{\bar{P}}$, $\bar{P} \in \mathrm{Path}(V^m, V^n)$, form a basis in $\mathrm{Hom}_{S_m}(V^m, V^n)$ & $J_i \varphi_{\bar{P}} = w_i \varphi_{\bar{P}}$ $\forall i = m+1, \dots, n$.

$m=1$, $P \in \mathrm{Path}(V^n) \rightsquigarrow v_P := \varphi_P \in V^n$, $w_P = (w_1, \dots w_n)$ w. $w_1 = 0$.

Corollary 2: The elements v_P form a basis in V^n w. $J_i v_P = w_i v_P$

Example: $V^n = \mathrm{refl}_n$, $P = (\mathrm{triv}_1 \to \mathrm{triv}_2 \to \mathrm{triv}_i \to \mathrm{refl}_{i+1} \to \dots \to \mathrm{refl}_n)$

$v_P = (1, \dots, 1, -i, 0, \dots 0)$, $w_P = (0, 1, \dots, i-1, -1, i, \dots, n-2)$.

Corollary 3: $\underline{P} \in \mathrm{Path}(V^m)$, $\bar{P} \in \mathrm{Path}(V^m, V^n) \rightsquigarrow P = \underline{P}\,\bar{P}$. Then v_P is proportional to $\varphi_{\bar{P}}(v_{\underline{P}})$.

1) Uniqueness of weights.

Thm: $P, P' \in Path_n$ & $w_P = w_{P'} \Rightarrow P = P'$.

Why do we care?

Def'n: $Wt_n = \{w_P \mid P \in Path_n\} \subset \mathbb{C}^n$. Say $w_P, w_{P'} \in Wt_n$ are r-equivalent if P, P' are paths to the same representation.

Thm implies

- $P \mapsto w_P : Path_n \xrightarrow{\sim} Wt_n$
- r-equivalence $\Leftrightarrow$ paths have the same end pt, so is equivalence.
- $Irr(\mathbb{C}S_n) \xrightarrow{\sim} Wt/\sim_r$ (equiv. classes for r-equivalence).
- if $V^n \in Irr(\mathbb{C}S_n)$ corresponds to some equiv. class, then have basis in V^n which is in bijection w. this equiv. class.

Task: describe Wt_n & the r-equivalence.

Fact (from [RT0]): A is assoc. algebra, V is fin. dim. irred. A-module. If $z \in A$ is central, then z acts on V by scalar.

Proof of Thm: induction on n.

- Base $n=1$: vacuous b/c have only one irrep, it has dim $=1$.
- Step: know claim for $n-1$.

$P, P' \in Path_n \rightsquigarrow$ truncations $\underline{P}, \underline{P}' \in Path_{n-1}$. If $w_P = (w_1, \ldots w_n)$

$\Rightarrow w_{\underline{P}} = (w_1, \dots, w_{n-1}) = w_{\underline{P}'} \Rightarrow \underline{P} = \underline{P}'$ (by ind. assumption)

Let $U \in Irr(\mathbb{C}S_{n-1})$ be end-pt for $\underline{P} = \underline{P}'$ & $V, V' \in Irr(\mathbb{C}S_n)$ – end pts for P, P'. Need to show $V \simeq V'$.

Claim: $\forall z \in Z_{n-1}(n)$, z acts on $U \subset V$ & $U \subset V'$ by scalars, $\chi(z), \chi'(z)$; Moreover $\chi(z) = \chi'(z)$.

Check Claim: $Z_{n-1}(n)$ is gen'd by $Z_{n-1}(n-1)$ & J_n. It's enough to check claim for generators.

- $z \in Z_{n-1}(n-1)$: use Fact for $A = \mathbb{C}S_{n-1}$, irred. module U $\Rightarrow \chi(z) = \chi'(z)$.
- $z = J_n$: $\chi(J_n) = w_n = w_n' = \chi'(J_n)$.

– claim is checked.

Center $Z_n(n)$ of $\mathbb{C}S_n$ sits inside $Z_{n-1}(n) \Rightarrow \forall z \in Z_n(n)$, z acts on $U \subset V$, $U \subset V'$ by same scalar. By Fact, z acts on V by a scalar, $\chi_V(z)$, on V' by scalar, $\chi_{V'}(z)$. So $\chi_V(z) = \chi_{V'}(z)$ $\forall z \in$ center of $\mathbb{C}S_n$.

$\Rightarrow V \simeq V'$. Reason: $\mathbb{C}S_n = \bigoplus_{V \in Irr(\mathbb{C}S_n)} End_{\mathbb{C}}(V)$

$\Rightarrow$ center of $\mathbb{C}S_n = \bigoplus_V \mathbb{C} \cdot id_V$ & center acts on V via projection to $\mathbb{C}\, id_V$. These are different for non-isom. irreps. $\square$

2) Varying path.

Fix $P \in Path_n$ & i w. $1 \le i < n$

$Path(P,i) = \{ P' = (V'^1 \to V'^2 \to \dots \to V'^n) \mid V^j = V'^j \ \forall j \ne i \}$

Task: Understand $w_{P'}$ for $P' \in Path(P,i)$.

Theorem: $w_P = (w_1, \dots w_n)$. Then:

(1) $w_i \ne w_{i+1}$.

(2) if $w_{i+1} = w_i \pm 1$, then $Path(P,i) = \{P\}$.

(3) if $w_{i+1} \ne w_i \pm 1$, then $Path(P,i) = \{P, P'\}$ w $P \ne P'$ & $w_{P'}$ is obtained from w_P by swapping w_i, w_{i+1}.

(4) if $w_{i+1} = w_i \pm 1$ & $i < n-1 \Rightarrow w_{i+2} \ne w_i$.

– to be proved in Lec 4.

$V := V^n$, $V_{P,i} = Span_{\mathbb{C}}(v_{P'} \mid P' \in Path(P,i))$ so $v_{P'}$'s - basis in $V_{P,i}$.

Prop'n: $V_{P,i}$ is irreducible $Z_{i-1}(i+1)$-submodule in V.

Proof: $P = P_0 P_1 P_2$ (concatenation) w.

$P_0 \in Path(V^{i-1})$, $P_1 \in Path(V^{i-1}, V^{i+1})$, $P_2 \in Path(V^{i+1}, V^n)$

$Path(P,i) = \{ P_0 P_1' P_2 \mid P_1' \in Path(V^{i-1}, V^{i+1}) \}$

By Cor 3 in Sec 0, $v_{P'} = \varphi_{P_2}(\varphi_{P_1'}(v_{P_0}))$

Consider linear map

$(*)\quad Hom_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1}) \longrightarrow V, \ \varphi \mapsto \varphi_{P_2}(\varphi(v_{P_0}))$

$\varphi_{P_1'} \longmapsto v_{P'}$

basis in $V_{P,i}$

basis in $Hom_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1})$

so $(*)$ is isomorphism onto $V_{p,i}$.

Since $\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1})$ is irred. $Z_{i-1}(i+1)$-module, it remains to show $(*)$ is $Z_{i-1}(i+1)$-module:

φ_{P_2} is $\mathbb{C}S_{i+1}$-linear & $Z_{i-1}(i+1) \subset \mathbb{C}S_{i+1}$, so φ_{P_2} is $Z_{i-1}(i+1)$-linear. So it remains to show that

$$\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1}) \to V^{i+1},\ \varphi \mapsto \varphi(v_{P_0})$$ is

$Z_{i-1}(i+1)$-linear. This follows from (1) in Sec 0. $\square$

3) Degenerate affine Hecke algebra.

Goal: understand $Z_{i-1}(i+1)$

Generators: $Z_{i-1}(i-1)$ - central subalgebra, $J_i, J_{i+1}, (i, i+1)$

Want: relations between $J_i, J_{i+1}, (i, i+1)$.

Lemma 1:

(2) $J_i J_{i+1} = J_{i+1} J_i$, $(i, i+1)^2 = 1$, $(i, i+1) J_i = J_{i+1}(i, i+1) - 1$.

Proof - exercise (proved in the notes, Lem 4.1).

Def'n: $H(2)$ is the assoc. algebra w. generators X_1, X_2, T & relations:

(3) $X_1 X_2 = X_2 X_1$, $T^2 = 1$, $TX_1 = X_2 T - 1$.

So have unique alg. homom. $H(2) \to Z_{i-1}(i+1)$:

$X_1 \mapsto J_i$, $X_2 \mapsto J_{i+1}$, $T \mapsto (i, i+1)$.

So every $Z_{i-1}(i+1)$-module can be viewed as $H(2)$-module.

Lemma 2: Let M be irred $Z_{i-1}(i+1)$-module. Then it's also irreducible over $H(2)$.

Proof: $Z_{i-1}(i+1)$ is generated by:

- image of $H(2)$
- a central subalgebra $Z_{i-1}(i-1)$, which acts on $\forall$ irredible by scalars.

So $M' \subset M$, a subspace, is $Z_{i-1}(i-1)$-stable. So, M' is $Z_{i-1}(i+1)$-stable $\Leftrightarrow$ M' is $H(2)$-stable. $\square$