0) Recap
1) Irreducible representations of $H(2)$ & applications.
2) Completion of classification.
3) Standard Young tableaux vs bases.

0) Recall that to a path $P = (V^1 \to V^2 \to \ldots \to V^n)$ we assign a basis vector $v_P \in V^n$ & $J_i v_P = w_i v_P$, where $w_i$ is the scalar of action of $J_i$, $i=1,\ldots,n$, on $V^{i-1} \subseteq V^i$. We say that $w_P = (w_1, \ldots, w_n)$ is the weight of $P$. Recall: $w_1 = 0$.

**Thm 1:** $w_P = w_{P'} \Rightarrow P = P'$

Let $Wt_n$ denote the set of all weights (for $CS_n$-modules). For $w, w' \in Wt_n$ we write $w \sim w'$ (r-equivalent) if $w, w'$ correspond to the same $V^n \in \text{Irr}(CS_n)$. So $Wt_n / \sim \Rightarrow \text{Irr}(CS_n)$, and equiv. class $\leftrightarrow$ basis in the corresponding irrep.

Now fix $P = (V^1 \to V^2 \to \ldots \to V^n)$, $i < n$.

$\text{Path}(P, i) = \{ P' = (V'' \to V''' \to \ldots \to V^n) \mid V^j = V'' \land j \neq i \}$

$V_{P, i} = \text{Span}_C(v_P, \{ p' \in \text{Path}(P, i) \})$ - has basis $v_P$.

**Prop:** $V_{P, i} \subseteq V^n$ is an irreducible $Z_{i-1}$-submodule.
To understand irreducible \( Z_{i-1, (i+1)} \)-modules we've defined the degenerate affine Hecke algebra \( H(c) \) w. generators \( X, X, T \) & relations

(1) \( X X = X X, T^2 = 1, TX = X T - 1 \)

(2) \( X T = TX - 1 \)

\( \exists \) (unique) alg. homom. \( H(c) \to Z_{i-1, (i+1)} \), \( X_i \to j_i, X_{i+1} \to j_i, T \to (i, i+1) \).

And if \( M \) is an irreducible \( Z_{i-1, (i+1)} \)-module, then it remains irreducible as an \( H(c) \)-module.

Our main goal today is to prove

Thm 2: Let \( W_p = (w_1, w_2) \). Then

(1) \( w_i \neq w_{i+1} \)

(2) if \( w_i = w_i \pm 1 \), then \( \text{Path}(P, i) = \{P \} \)

(3) if \( w_i \neq w_i \pm 1 \), then \( \text{Path}(P, i) = \{P, P'\} \) w. \( P \neq P' \) & \( W_p = (w_1, w_2, w_3, \ldots, w_n) \).

(4) if \( w_i = w_i \pm 1 \) & \( i < n-1 \), then \( w_{i+1} \neq w_i \).

Rem (used in HW): for \( d \geq 1 \), have \( H(d) \) w.

- generators \( X_1, \ldots, X_d, T_1, \ldots, T_{d-1} \)
- relations: \( X_i X_j = X_j X_i \)
  
  \( T_i^2 = 1, T_i T_j = T_j T_i \) if \( |i-j| > 1, T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \)

exactly the relations for \( (1, 2), \ldots, (n-1, n) \) in \( S_n \)

\( T_i X_i = X_i T_i - 1, T_i X_j = X_j T_i \) \( \neq i, \neq j \neq i, i+1 \).

\( \exists \) homom \( H(d) \to Z_c (l + d) \), \( X_i \to j_{i+1}, T_i \to (l, l+1) \).
1) Irreducible $H(2)$-modules & applications

1.1) Classification

$M$: finite dim, irred. $H(2)$-module.

$X_1X_2 = X_2X_1 \Rightarrow \exists \ m \in M \mid X_1m = am, X_2m = bm \ (a, b \in C)$

Case 1: $Tm \in Cm \Rightarrow T^2 = 1 \Rightarrow$

Case 1.1: $Tm = m$: apply $TX_1 = X_2T - 1$ to $m$:

$am = 6m - m \Rightarrow 6 = a + 1.$

or

Case 1.2: $Tm = -m \Rightarrow 6 = a - 1.$

Case 2: $Tm \notin Cm \ (m \ & Tm \ \text{lin. indep.'t})$,

$X_1Tm = [X_1T = TX_2 - 1] = TX_1m - m = [X_2m = 6m] = 6(Tm) - m$

$X_2Tm = [X_2T = TX_1 + 1] = a(Tm) + m$

$T(Tm) = m$

So $\text{Span}(m, Tm)$ is $H(2)$-submodule $\Rightarrow M = \text{Span}(m, Tm)$.

In the basis $m, Tm$ of $M$ generators act $T$:

$T \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ X_1 \mapsto \begin{pmatrix} a & 0 \\ -1 & b \end{pmatrix}, \ X_2 \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

(3)

These matrices satisfy rels. (1) & $a, b \in C$. So (3) defines

$H(2)$-module $M(a, b)$ on $C^2$ to be denoted by $M(a, b)$.

Lemma: (1) $M(a, b)$ is irreducible $\iff 6 \neq a \pm 1.$

(2) if $6 \neq a \pm 1 \ & M(a', b') = M(a, b)$, then either $(a', b') = (a, b)$ or $(a', b') = (b, a)$.
Proof: (1): Assume \( b \neq a \). Then \( X_1, X_2 \) are diagonalizable so have two different eigenvectors. The subspace of \( M \) stable under \( X_1, X_2 \) is spanned by some eigenvectors. For eigenvector \( m' \), \( Cm' \) is \( T \)-stable \( \Rightarrow b = a \pm 1 \). Conversely, if \( b = a \pm 1 \), then \( m \pm Tm \) is eigenvector \( \Rightarrow M(\pm b) \) is not irreducible.

If \( b = a \), then \( X_1, X_2 \) are not diagonalizable, this case is 

\[ \text{Exercise.} \]

(2): Assume \( b \neq a \). Then \( M(\pm b) \) has another eigenvector \( w \) with \( e \)-values \( b \) for \( X_1 \) and \( a \) for \( X_2 \). So \( M(b,a) \sim M(\pm b) \).

If \( (a', b') \neq (a, b) \) or \( (b, a) \) \( \Rightarrow M(c, b') \neq M(\pm b, a) \), \( b' \neq a' \) is not an eigenvalue of \( X_1 \) on \( M(\pm b) \).

The case \( b = a \): \text{exercise} \[ \square \]

We've proved:

Prop: Finite dim'l irred \( H(2) \)-modules are classified by

\( (a, b) \in \mathbb{C}^2 : (a, b) \mapsto L(a, b) \) \( w. \ L(a, b) \sim L(b, a) \) if \( b \neq a \pm 1 \).

Moreover:

\[ \begin{align*}
& \bullet (a, b) \text{ is a pair of eigenvalues of } X_1, X_2 \text{ on } L(a, b) \\
& \bullet \text{if } b = a + 1, \text{ then } L(a, b) = \mathbb{C} \text{ with } T \mapsto 1, X_1 \mapsto a, X_2 \mapsto b. \\
& \bullet \text{if } b = a - 1, \text{ then } L(a, b) = \mathbb{C}, \ T \mapsto -1, X_1 \mapsto a, X_2 \mapsto b. \\
& \bullet \text{if } b \neq a \pm 1, \text{ then } L(a, b) \sim M(a, b) \text{ from eq. (3)} \\
& \bullet X_1, X_2 \text{ act on } L(a, b) \text{ diagonalizably } \Rightarrow a \neq b.
\]
1.2) Consequence:

**Thm 2:** Let \( W_p = (w_1, \ldots, w_n) \). Then

1. \( W_i \neq W_{i+1} \)
2. If \( W_{i+1} = W_i + 1 \), then Path\((P_i)\) = \( \{P, P'\} \) \( W_p = (w_1, \ldots, w_{i-1}, W_i, W_i, W_i, \ldots, W_n) \).
3. If \( W_{i+1} \neq W_i + 1 \), then Path\((P_i)\) = \( \{P, P'\} \) \( W_p = (w_1, \ldots, w_{i-1}, W_i, W_i, W_i, \ldots, W_n) \).
4. If \( W_{i+1} = W_i + 1 \) & \( i < n-1 \), then \( W_{i+2} = W_i \).

**Proof:** \( W_p = (W'_1, \ldots, W'_n) \) for \( P' \in \text{Path}(P_i) \)

- \( W'_j \) depends only on \( V^j \) & \( V^j \) coincide w. \( V^{j-1}, V^j \) if \( j \neq i, i+1 \Rightarrow W'_j = W_j \).
- \( (W'_i, W'_{i+1}) \) - simul. eigenvalues for \( J_i, J_{i+1} \in \mathbb{Z}_{c_{i+1}}(i+1) \), i.e. for \( X, X_e \in \mathbb{H}(2) \). Moreover, \( V_{P_i} \) is irreducible \( \mathbb{H}(2) \)-module, where \( X, X_e \) act diagonalizably.

Use Prop'n: get (1)-(3) of Thm.

**Proof of (4):** Assume \( W_i = W_{i+1} = W_{i+2} \). Then \( C_{W_p} \) is stable under \( \mathbb{Z}_{c_{i+1}}(i+1) \) & \( \mathbb{Z}_{c_{i+2}}(i+2) \). So \( (i, i+1)W_p = \pm W_p \) & \( (i+1, i+2)W_p = \mp W_p \).

Observe:

\[
(i, i+1)(i+1, i+2)(i, i+1) = (i, i+2) = (i+1, i+2)(i, i+1)(i+1, i+2)
\]

Apply the equality to \( V_p : \mp W_p = \pm W_p \). Contradiction \( \square \)

2) Completion of classification of \( \text{Irr}(C_5) \).

**Defn:** An admissible permutation of \( W \in C^n \) is \( W = (w_1, \ldots, w_n) \)

\[
\mapsto (w_1, \ldots, w_{i-1}, W_i, W_i, w_{i+1}, \ldots, W_n)
\]

for \( i \) w. \( w_i \neq W_i \).

- Two edits of \( C^n \) are \( c \)-equivalent if one is obtained
from the other by a sequence of admissible permutations. This is an equiv.

\[ \sim_c \]

\[ \cdot \text{we } G^n \text{ is a combinatorial weight if } \forall \text{ c-equiv } w' \text{ have:} \]

\[ - w'_i = 0 \]

\[ - w'_i \neq w'_i \]

\[ - \text{if } w'_{i+1} = w'_i \pm 1 \text{ } \& \text{ } i < n-1 \Rightarrow w'_{i+2} \neq w'_i \]

\[ \cdot \text{ch}_{W_n} = \text{set of all comb. wts.} \]

**Corollary of Thm 2:**

1. \[ W_t_n < \text{ch}_{W_t} \& \text{ } W_t_n \text{ is union of c-equiv classes.} \]
2. \[ \sim_c \Rightarrow \sim_r \]

\[ |\text{spatns of } n \mathbb{Z}| = |\text{Irr}(CS_n)| = |W_t_n / \sim_r| \]

\[ |W_t_n / \sim_c| \]

\[ \text{Lemma: in every c-equiv class in ch}_{W_t} \text{ we have representative:} \]

\[ (0, 1, ..., n_1, -1, 0, ..., n_2 - 2, -2, ..., n_3 - 3, ..., -k, ..., n_k - k) \text{ for } n_i \geq n_2 \geq ... \geq n_k \text{ } w \text{ } n_i + n_2 + ... + n_k = n. \]

\[ \text{Proof: see notes - look at the max. elt in lexicographic order} \]

\[ \text{By Lemma, all inequalities in (4) are actually equalities.} \]
Therefore:

\[ W_t = ch_t \]

\[ \sim_c = \sim_r \]

\[ (n_1 ... n_k) \] in Lemma are uniquely determined by \( c \)-equiv class.

\[ \text{so } \text{Irr}(CS_n) \sim \{ \text{partitions of } n \} \]

\[ W_t / \sim_r \]

\textbf{Ex: } \( n = 4 \):

\text{triv}_4: \text{unique wt } (0,1,2,3) \text{ - in the form of Lemma: part } n = 4 \).

\text{sgn}_4: (0,1,2,3) - \cdot - \cdot - \cdot \cdot (1^4)

\text{refl}_4: 3 \text{ wts: } (0,1,1,2), (0,1,1,2), (0,1,2,1) \rightarrow (3,1) \text{ is as in Lemma}

\text{C}^2: 2 \text{ wts } (0,1,1,0), (0,1,0) \text{ as in Lemma}

3) For part \( n \lambda \), let \( V_{\lambda} \in \text{Irr}(CS_n) \) corresp. irrep. A basis in \( V_{\lambda} \) is labelled by the combinatorial wts that correspond to \( \lambda \). Those are in bijection w. the standard Young tableaux of shape \( \lambda \). Section 5.2 in the notes.