Representations of symmetric groups, part 4.

0) Recap.
1) Irreducible representations of $\mathcal{H}(2)$ & applications.
2) Completion of classification.
3) Standard Young tableaux vs bases.

0) Recall that to a path $P=(V^1\to V^2\to\ldots\to V^n)$ we assign a basis vector $v_P\in V^n$ & $\mathcal{J}_i v_P = w_i v_P$, where w_i is the scalar of action of $\mathcal{J}_i$, $i=1,\ldots n$, on $V^{i-1}\subset V^i$. We say that $w_P=(w_1,\ldots w_n)$ is the **weight** of P. Recall: $w_1=0$.

Thm 1: $w_P=w_{P'} \Rightarrow P=P'$.

Let Wt_n denote the set of all weights (for $\mathbb{C}S_n$-modules). For $w,w'\in Wt_n$ we write $w\sim_r w'$ (r-equivalent) if w,w' correspond to the same $V^n\in Irr(\mathbb{C}S_n)$. So $Wt_n/\sim_r \xrightarrow{\sim} Irr(\mathbb{C}S_n)$, $\forall$
$\forall$ equiv. class $\xleftrightarrow{\sim}$ basis in the corresponding irrep.

Now fix $P=(V^1\to V^2\to\ldots\to V^n)$, $i<n \rightsquigarrow$

$$Path(P,i)=\{P'=(V'^1\to V'^2\to\ldots\to V'^n) \mid V^j=V'^j\ \forall j\neq i\}$$

$$V_{P,i}=Span_{\mathbb{C}}(v_{P'} \mid P'\in Path(P,i)) \text{ – has basis } v_{P'}.$$

Prop: $V_{P,i}\subset V^n$ is an irreducible $\mathcal{Z}_{i-1}(i+1)$-submodule.

To understand irreducible $\mathcal{Z}_{i-1}(i+1)$-modules we've defined the degenerate affine Hecke algebra $\mathcal{H}(2)$ w. generators X_1, X_2, T & relations

(1) $X_1X_2 = X_2X_1$, $T^2 = 1$, $TX_1 = X_2T - 1$
$\Downarrow$
(2) $X_1T = TX_2 - 1$

$\exists$ (unique) alg. homom'm $\mathcal{H}(2) \longrightarrow \mathcal{Z}_{i-1}(i+1)$, $X_1 \mapsto J_i$, $X_2 \to J_{i+1}$, $T \mapsto (i,i+1)$.
And if M is an irreducible $\mathcal{Z}_{i-1}(i+1)$-module, then it remains irreducible as an $\mathcal{H}(2)$-module.

Our main goal today is to prove

Thm 2: Let $w_P = (w_1, \ldots w_n)$. Then

(1) $w_i \neq w_{i+1}$

(2) if $w_{i+1} = w_i \pm 1$, then $Path(P,i) = \{P\}$

(3) if $w_{i+1} \neq w_i \pm 1$, then $Path(P,i) = \{P, P'\}$ w. $P' \neq P$ &
$w_{P'} = (w_1, \ldots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \ldots w_n)$.

(4) if $w_{i+1} = w_i \pm 1$ & $i < n-1$, then $w_{i+2} \neq w_i$.

Rem (used in HW): for $d \geq 1$, have $\mathcal{H}(d)$ w.

- generators $X_1, \ldots X_d$, $T_1, \ldots, T_{d-1}$
- relations: $X_iX_j = X_jX_i$

$$T_i^2 = 1,\ T_iT_j = T_jT_i \text{ if } |i-j|>1,\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$$

–exactly the relations for $(1,2), \ldots, (n-1,n)$ in S_n

$$T_iX_i = X_{i+1}T_i - 1,\ T_iX_j = X_jT_i \quad \forall i, \forall j \neq i, i+1.$$

$\exists$ homom $\mathcal{H}(d) \longrightarrow \mathcal{Z}_\ell(\ell+d)$, $X_i \mapsto J_{\ell+i}$, $T_i \mapsto (\ell+i, \ell+i+1)$.

1) Irreducible $H(2)$-modules & applications.

1.1) Classification.

M : finite dim'l irred. $H(2)$-module.

$X_1X_2=X_2X_1 \Rightarrow \exists\ m\in M \mid X_1 m=am,\ X_2m=bm\ (a,b\in\mathbb{C})$

Case 1: $Tm\in\mathbb{C}m$: $T^2=1 \Rightarrow$

Case 1.1: $Tm=m$: apply $TX_1=X_2T-1$ to m:

$am=bm-m \Rightarrow b=a+1.$

or

Case 1.2: $Tm=-m \Rightarrow b=a-1.$

Case 2: $Tm\notin\mathbb{C}m$ (m & Tm lin. indep't)

$X_1Tm=[X_1T=TX_2-1]=TX_2m-m=[X_2m=bm]=b(Tm)-m$

$X_2Tm=[X_2T=TX_1+1]\ \ldots\ =a(Tm)+m$

$T(Tm)=m$

So $\operatorname{Span}(m,Tm)$ is $H(2)$-submodule $\Rightarrow M=\operatorname{Span}(m,Tm)$.

In the basis m, Tm of M generators act by:

$$T\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix},\ X_1\mapsto\begin{pmatrix}a&0\\-1&b\end{pmatrix},\ X_2\mapsto\begin{pmatrix}b&0\\1&a\end{pmatrix}\qquad(3)$$

These matrices satisfy rel'ns (1) $\forall\, a,b\in\mathbb{C}$. So (3) defines $H(2)$-module str're on $\mathbb{C}^2$ to be denoted by $M(a,b)$.

Lemma: (1) $M(a,b)$ is irreducible $\Leftrightarrow b\neq a\pm1$.

(2) if $b\neq a\pm 1$ & $M(a',b')\cong M(a,b)$, then either $(a',b')=(a,b)$ or $(a',b')=(b,a)$.

Proof: (1): • Assume $b \neq a$. Then X_1, X_2 are diagonalizable so have two diff't eigenvectors. $\forall$ subspace of M stable under X_1, X_2 is spanned by some eigenvectors. For eigenvector m', $\mathbb{C}m'$ is T-stable $\Rightarrow b = a \pm 1$. Conversely, if $b = a \pm 1$, then $m \pm Tm$ is eigenvector $\Rightarrow M(a,b)$ is not irreducible.

• If $b = a$, then X_1, X_2 are not diagonalizable, this case is exercise.

(2): Assume $b \neq a$. Then $M(a,b)$ has another eigenvector w. e-values b for X_1 and a for X_2. So $M(b,a) \simeq M(a,b)$. If $(a',b') \neq (a,b)$ or $(b,a) \Rightarrow M(a',b') \not\simeq M(a,b)$, b/c a' is not eigenvalue of X_1 on $M(a,b)$.

The case $b = a$: exercise □

We've proved:

Prop: Finite dim'l irred $\mathcal{H}(2)$-modules are classified by $(a,b) \in \mathbb{C}^2$: $(a,b) \mapsto L(a,b)$ w. $L(a,b) \simeq L(b,a)$ if $b \neq a \pm 1$.

Moreover: • (a,b) is a pair of eigenvalues of X_1, X_2 on $L(a,b)$

• if $b = a+1$, then $L(a,b) = \mathbb{C}$ w. $T \mapsto 1$, $X_1 \mapsto a$, $X_2 \mapsto b$.

• if $b = a-1$, then $L(a,b) = \mathbb{C}$, $T \mapsto -1$, $X_1 \mapsto a$, $X_2 \mapsto b$.

• if $b \neq a \pm 1$, then $L(a,b) \simeq M(a,b)$ from eq. (3)

• X_1, X_2 act on $L(a,b)$ diagonalizably $\Leftrightarrow a \neq b$.

1.2) Consequence:

Thm 2: Let $w_P=(w_1,\dots w_n)$. Then

(1) $w_i\neq w_{i+1}$

(2) if $w_{i+1}=w_i\pm1$, then $Path(P,i)=\{P\}$

(3) if $w_{i+1}\neq w_i\pm1$, then $Path(P,i)=\{P,P'\}$ w. $P'\neq P$ &

$w_{P'}=(w_1,\dots,w_{i-1},w_{i+1},w_i,w_{i+2},\dots w_n)$.

(4) if $w_{i+1}=w_i\pm1$ & $i<n-1$, then $w_{i+2}\neq w_i$.

Proof: $w_{P'}=(w_1',\dots w_n')$ for $P'\in Path(P,i)$

- w_j' depends only on V'^{j-1} & V'^j – coincide w. V^{j-1}, V^j if $j\neq i,i+1 \Rightarrow w_j'=w_j$
- (w_i',w_{i+1}') – simul. eigenvalues for $J_i, J_{i+1}\in Z_{i-1}(i+1)$, i.e. for $X_1,X_2\in\mathcal{H}(2)$. Moreover, $V_{P,i}$ is irred'le $\mathcal{H}(2)$-module, where X_1,X_2 act diagonalizably.

Use Prop'n: get (1)-(3) of Thm.

Proof of (4): Assume $w_i=w_{i+2}=w_{i+1}\pm1$. Then $\mathbb{C}v_P$ is stable under $Z_{i-1}(i+1)$ & $Z_i(i+2)$. So $(i,i+1)v_P=\pm v_P$, $(i+1,i+2)v_P=\mp v_P$.

Observe:

$$(i,i+1)(i+1,i+2)(i,i+1)=(i,i+2)=(i+1,i+2)(i,i+1)(i+1,i+2)$$

Apply the equality to v_P: $\mp v_P=\pm v_P$. Contradiction $\square$

2) Completion of classification of $Irr(\mathbb{C}S_n)$.

Def'n: • An admissible permutation of $w\in\mathbb{C}^n$ is $w=(w_1,\dots w_n)\mapsto(w_1,\dots w_{i-1},w_{i+1},w_i,w_{i+2},\dots w_n)$ for i w. $w_{i+1}\neq w_i\pm1$.

• Two el'ts of $\mathbb{C}^n$ are c-equivalent if one is obtained from the other by a sequence of admissible permutations. This is an equiv. rel'n, $\sim_c$.

- $w \in \mathbb{C}^n$ is a combinatorial weight if $\forall$ c-equiv't w' have:
  - $w_1' = 0$
  - $w_i' \neq w_{i+1}'$
  - if $w_{i+1}' = w_i' \pm 1$ & $i < n-1 \Rightarrow w_{i+2}' \neq w_i'$.
- cWt_n = set of all comb. wts.

Corollary of Thm 2:

(1) $Wt_n \subset cWt_n$ & Wt_n is union of c-equiv. classes.

(2) $\sim_c \Rightarrow \sim_r$

$$
\begin{aligned}
|\{\text{part'ns of } n\}| = |Irr(\mathbb{C}S_n)| = &\ |Wt_n / \sim_r| \\
&\quad \wedge\!| \\
&\ |Wt_n / \sim_c| \\
&\quad \wedge\!| \\
&\ |cWt_n / \sim_c|
\end{aligned}
\qquad (4)
$$

Lemma: in every c-equiv. class in cWt_n we have representative:

$(0, 1, \ldots, n_1 - 1, -1, 0, \ldots n_2 - 2, -2, \ldots, n_3 - 3, \ldots, -k, \ldots n_k - k)$ for

$n_1 \geq n_2 \geq \ldots \geq n_k$ w $n_1 + n_2 + \ldots + n_k = n$.

Proof: see notes — look at the max. el't in lexicographic order.

By Lemma, all inequalities in (4) are actually equalities.

Therefore:
- $Wt_n = cWt_n$.
- $\sim_c = \sim_r$
- $(n_1, \dots n_k)$ in Lemma are uniquely determined by c-equiv. class.
- so $\mathrm{Irr}(\mathbb{C}S_n) \xrightarrow{\sim} \{\text{partitions of } n\}$

$$\begin{tikzcd}[column sep=small] \mathrm{Irr}(\mathbb{C}S_n) \arrow[rr, "\sim"] \arrow[dr, "\wr"] & & \{\text{partitions of } n\} \\ & Wt_n/\sim_r \arrow[ur, "\wr"] & \end{tikzcd}$$

Ex: $n=4$:

$triv_4$: unique wt $(0,1,2,3)$ – in the form of Lemma: part'n $=(4)$.

sgn_4: $(0,-1,-2,-3)$ — · — · — · — · — · — · — · - · (1^4)

$refl_4$: 3 wts: $(0,-1,1,2)$, $(0,1,-1,2)$, $\underbrace{(0,1,2,-1)}_{\text{is as in Lemma}}$ $\rightsquigarrow$ $(3,1)$

$\mathbb{C}^2$: 2 wts $(0,-1,1,0)$, $\underbrace{(0,1,-1,0)}_{\text{as in Lemma}}$ $(2,2)$

3) For part'n λ, let $V_\lambda \in \mathrm{Irr}(\mathbb{C}S_n)$ corresp. irrep. A basis in V_λ is labelled by the combinatorial wts that correspond to λ. Those are in bijection w. the standard Young tableaux of shape λ. Section 5.2 in the notes.