Representations of algebraic group & Lie algebras, I

0) Recap of bits of Algebraic geometry

1) Algebraic groups

Modifications in Sec. 1.3 on 02/12

0) Let $\mathbb{F}$ be an algebraically closed field

Definition: • By an embedded affine variety we mean a subset of $\mathbb{F}^n$ (for some n) defined by polynomial equations.

• Let $X\subset \mathbb{F}^n$, $Y\subset \mathbb{F}^m$ be embedded affine varieties. A map $\varphi: X\to Y$ is called polynomial (a.k.a. a morphism) if it's a restriction of a map $\mathbb{F}^n \to \mathbb{F}^m$ given by polynomials.

• The algebra of polynomial functions $\mathbb{F}[X]$ consists of polynomial maps $X\to\mathbb{F}$ w. usual addition and multiplication of functions.

Facts: i) The set $I(X)=\{f\in \mathbb{F}[x_1,..x_n] \mid f|_X=0\}$ is a radical ideal in $\mathbb{F}[x_1,..x_n]$ ($I(X)=\sqrt{I(X)}$). The assignment $X\mapsto I(X)$ gives a bijection between embedded affine varieties in $\mathbb{F}^n$ and radical ideals of $\mathbb{F}[x_1,..x_n]$ (Nullstellensatz). Moreover, $\mathbb{F}[X] \xrightarrow{\sim} \mathbb{F}[x_1,...,x_n]/I(X)$. It's a finitely generated $\mathbb{F}$-algebra w. a distinguished collection of generators: $x_i+I(X)$, $i=1,..n$. Moreover, the algebra $\mathbb{F}[X]$ contains no (nonzero) nilpotent elements.

ii) Let $\varphi: X\to Y$ be a morphism and let $f_1,...f_n \in \mathbb{F}[x_1,...,x_n]$ be such that $\varphi=(f_1,...f_n)|_X$. We get a homomorphism $\varphi^*: \mathbb{F}[Y]\to\mathbb{F}[X]$, $g\mapsto g\circ\varphi$. It sends $y_i+I(Y)$ to $f_i+I(X)$,

$i=1,\dots m$. The assignment $\varphi \mapsto \varphi^*$ defines a bijection between morphisms $X \to Y$ and algebra homomorphisms $\mathbb{F}[Y] \to \mathbb{F}[X]$. Moreover, this assignment is functorial: $(id_X)^* = id_{\mathbb{F}[X]}$ & for $\varphi: X \to Y$, $\psi: Y \to Z$, have $(\psi\varphi)^* = \varphi^* \psi^*$.

ii) allows us to talk about "abstract" affine varieties, X. They correspond to fin. generated $\mathbb{F}$-algebras w/o nilpotent elements. The choice of generators corresponds to an embedding of X into some $\mathbb{F}^n$ but we view X irrespective of an embedding. The notion of a morphism still makes sense in this setting.

Here are two useful constructions:

(i) Let X be an affine variety & $f \in \mathbb{F}[X]$. Then $X_f := \{x \in X \mid f(x) \neq 0\}$ is an affine variety w. $\mathbb{F}[X_f] = \mathbb{F}[X][f^{-1}]$.

(ii) Let X, Y be affine varieties. Then $X \times Y$ is also an affine variety w. $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y] \xrightarrow{\sim} \mathbb{F}[X \times Y]$, $[f \otimes g](x,y) := f(x)g(y)$.

Rem: Subsets in affine variety X defined by polynomial equations are called Zariski closed, they are indeed closed subsets in a topology, the Zariski topology. A subset in X is called Zariski open if its complement is Zariski closed. Note that a Zariski closed subset, say, Y, of X is again an affine variety (this may fail for open subvarieties). The homomorphism $i^*: \mathbb{F}[X] \to \mathbb{F}[Y]$ corresp. to $i: Y \hookrightarrow X$ is surjective.

1) Algebraic groups.

1.1) Definition & examples.

Consider the group $GL_n(\mathbb{F})$ of all nondegenerate $n\times n$-matrices w. coefficients in $\mathbb{F}$. Equivalently, if V is an n-dim'l vector space over $\mathbb{F}$, then choosing a basis in V we identify $GL(V)$ w. $GL_n(\mathbb{F})$.

Note that $GL_n(\mathbb{F}) = \{A\in Mat_n(\mathbb{F}) \mid \det(A)\neq 0\}$. So $GL_n(\mathbb{F})$ is an affine variety w. $\mathbb{F}[GL_n(\mathbb{F})] = \mathbb{F}[x_{ij}][\det^{-1}]$, where x_{ij}, $i,j=1,\ldots n$, are matrix coefficients. See (i) in Sec. 0.

Definition: By an algebraic group we mean a subgroup of some $GL_n(\mathbb{F})$ that is Zariski closed, i.e. given by polynomial equations.

Examples of algebraic groups.

0) $GL_n(\mathbb{F})$.

1) $SL_n(\mathbb{F}) = \{A\in GL_n(\mathbb{F}) \mid \det(A)=1\}$, a single polynomial equation. This is the special linear group.

2) Assume $\operatorname{char}\mathbb{F}\neq 2$. Set $O_n(\mathbb{F}) = \{A\in GL_n(\mathbb{F}) \mid AA^T = I\}$. More conceptually, let B be a non-degenerate symmetric form on a vector space V of $\dim = n$ (all these forms have an orthonormal basis so there's no difference between them). Then we can consider $O(V,B) := \{g\in GL(V) \mid B(gu,gv)=B(u,v)\ \forall u,v\in V\}$. A choice of an orthonormal basis for B identifies $O(V,B)$ w. $O_n(\mathbb{F})$. The group $O_n(\mathbb{F})$ (or $O(V,B)$) is called the orthogonal group.

Note that $\det(A)=\pm 1$ for $A\in O_n(\mathbb{F})$. Set $SO_n(\mathbb{F}) :=$

$= \{A \in O_n(\mathbb{F}) \mid \det A = 1\}$. This is also an algebraic group, the special orthogonal group.

3) Similarly, for a non-degenerate skew-symmetric form ω on a finite dimensional vector space V (then, automatically, $\dim V$ is even) we can similarly consider the symplectic group $Sp(V,\omega) = \{g \in GL(V) \mid \omega(gu, gv) = \omega(u,v)\ \forall\, u,v \in V\}$. One can find a basis $v_1, \dots, v_{2n} \in V$ s.t $\omega(v_i, v_j) = \pm \delta_{i+j, 2n+1}$, where we have a "+" $\Leftrightarrow i \leq n$. Let J be the matrix of ω in this basis:
$J = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & 0 \end{pmatrix}$ so that $Sp(V,\omega) \xrightarrow{\sim} \{A \in GL_{2n}(\mathbb{F}) \mid A^T J A = J\} =: Sp_{2n}(\mathbb{F})$.

The groups in Examples 1-3 are called classical. They are extremely important.

4) The subgroups of upper-triangular, $\left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$, upper-uni-triangular, $\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}$, and diagonal, $\{\mathrm{diag}(z_1, \dots, z_n)\}$ matrices in $GL_n(\mathbb{F})$ are algebraic.

5) The multiplicative group $\mathbb{F}^\times = GL_1(\mathbb{F})$ often denoted by $\mathbb{G}_m$, and the additive group $\mathbb{F} = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \subset GL_2(\mathbb{F})$ often denoted by $\mathbb{G}_a$ are algebraic.

Exercise: If G_1, G_2 are algebraic groups, then so is their product (hint: $GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F})$ embeds into $GL_{n_1+n_2}(\mathbb{F})$ as the subgroup of block-diagonal matrices).

Rem: Note that every algebraic group in our sense, G, is Zariski closed in an affine variety, $GL_n(\mathbb{F})$, hence is an affine variety itself. Moreover, note that the multiplication map $GL_n(\mathbb{F}) \times GL_n(\mathbb{F}) \longrightarrow GL_n(\mathbb{F})$ and the inversion map $GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ are given by polynomials in the matrix coefficients (and also $\det^{-1}$ for the latter) so are morphisms. It follows that (exercise)

(*) G is an affine variety & the multiplication $G \times G \to G$ & the inversion, $G \to G$, maps are morphisms.

We can take (*) for a (more conceptual) definition of an algebraic group, however we get the same objects: every G satisfying (*) embeds as a Zariski closed subgroup into some $GL_n(\mathbb{F})$ (see, e.g. Thm 8 in §3.1.6 of [OV]).

Rem: (*) is parallel to the definition of a Lie group (replace C^∞-manifolds there with affine algebraic varieties). One can show that every algebraic group is smooth as a variety (try to prove this if you know what "smooth" means - use that the action of the (variety) automorphism group on G is transitive). It follows that for $\mathbb{F} = \mathbb{C}$, an algebraic group is a complex (analytic) Lie group.

1.2) Homomorphisms & representations

Definition: Let G, H be algebraic groups.

· By an (algebraic group) homomorphism $G \to H$ we mean a group homomorphism that is also a morphism of varieties.

· Let V be a finite dimensional space. By a rational representation of G in V we mean an algebraic group homomorphism $G \to GL(V)$ (we'll elaborate on why "rational" later).

In other words, a rational representation of G is one with matrix coefficients in $\mathbb{F}[G]$.

Example 1: i) The groups $GL_n(\mathbb{F}), SL_n(\mathbb{F}), O_n(\mathbb{F}), Sp_n(\mathbb{F})$ (n is even in the last case) are embedded into $GL_n(\mathbb{F})$, hence come with a rational representation in $V=\mathbb{F}^n$ called tautological.

ii) If V is a rational representation of G, then so are V^* and sub- and quotient representations of V. This is left as an exercise - look at matrix coefficients.

iii) If V, W are rational representations of G, then so are $V \oplus W$ & $V \otimes W$, exercise.

Example 2: Suppose char $\mathbb{F}=p>0$. In this case the map $x \mapsto x^p$ is an automorphism of $\mathbb{F}$ (the Frobenius automorphism). The map $Fr: GL_n(\mathbb{F}) \to GL_n(\mathbb{F}), (a_{ij}) \mapsto (a_{ij}^p)$ is therefore an algebraic group homomorphism. It's an automorphism of an abstract group but not of an algebraic group (pth root is not a polynomial).

Now let $G \subset GL_n(\mathbb{F})$ be defined by polynomials with coefficients in $\mathbb{F}_p$ (and not just in $\mathbb{F}$). This is the case in Ex.

1-4 of Sec 1.1. Then Fr restricts to G ($\forall f \in \mathbb{F}_p[x_{ij}]$ we have $f(a_{ij}) = 0 \Leftrightarrow f(a_{ij}^p) = 0$ b/c $x \mapsto x^p$ is id on $\mathbb{F}_p$) so we get the algebraic group Frobenius homomorphism $Fr: G \to G$, an abstract group automorphism. It plays a very important role in the study of rational representations of G.

Rem (on terminology): for the group $G = GL_m(\mathbb{F})$, a representation is "rational" means: matrix coefficients are polynomials in the matrix entries, x_{ij}, & $\det^{-1}$. One also considers "polynomial" representations - those, were matrix coefficients are polynomials just in x_{ij}'s. For example, the tautological representation, its tensor powers, etc. are polynomial, while its dual is not polynomial.

1.3) Big picture & connections.

As a part of the general ideology, we care about the structure and representation theory of "simple" algebraic groups & their relatives ("semisimple" & "reductive") groups.

Definition: An algebraic group G is simple if it is connected (in the Zariski topology) & all normal algebraic subgroups of G are finite. We also require G is not commutative.

For example, $SL_n(\mathbb{F})$, $n \geqslant 2$, $SO_n(\mathbb{F})$, $n = 3$ or $n \geqslant 5$ (SO_4 is "semisimple" but not simple), $Sp_{2n}(\mathbb{F})$, $n \geqslant 1$, are simple. In a way, there are just five more examples, the exceptional groups: G_2, F_4, E_6, E_7, E_8. We'll discuss more on that later.

Simple algebraic groups give the most important kind of symmetry in Mathematics. They are also the most central object in Representation theory - with a few exceptions confirming the rule everything considered in Representation theory is related to simple algebraic groups in one way or another - for example, S_n appears in at least three ways when we study SL_n & its representations.

One manifestation of this central role is a connection to finite simple groups. Let G be a simple algebraic group over $\mathbb{F}=\overline{\mathbb{F}}_p$. As in Example 1 in Section 1.1, G embeds into some $GL_n(\mathbb{F})$ as a subgroup defined by polynomial equations w. coefficients in $\mathbb{F}_p$. So, by Example 2 of Section 1.2, we get the Frobenius endomorphism $Fr: G \to G$. Pick $k \geqslant 1$ and set $\varphi := Fr^k$. Let $G^{\varphi} \subset GL_n(\mathbb{F})^{\varphi}$ be the fixed point groups. Note that $GL_n(\mathbb{F})^{\varphi} = [\varphi \text{ acts entry-wise}] = GL_n(\mathbb{F}_q)$. In particular, G^{φ} is a finite group, e.g. for $G = SL_n(\mathbb{F})$ get $G^{\varphi} = SL_n(\mathbb{F}_q)$.

These groups are "almost simple" - we can produce finite simple groups out of them. This construction can be generalized - one can replace Fr^k w. its "twisted versions" - we'll mention this later in the course. As was mentioned in Lec 1, most finite simple groups are produced in this way.