Representations of algebraic group & Lie algebras, 2.5.

1) Hopf algebras

2) Distribution algebras, pt. 1.

1) Questions about affine varieties (or schemes) usually get translated to the language of algebras of functions. So, we can ask: how is an algebraic group structure on G reflected on its algebra of functions.

To have an algebraic group structure means:

- we have the multiplication morphism $m: G\times G \to G$
- we have the unit element that can be viewed as a morphism $\mathbb{1}: pt \to G$
- we have the inversion morphism $\iota: G \to G$.

The axioms m, ε, ι should satisfy are exactly those of a group, i.e.

- m is associative, equivalently we have the following commutative diagram

$$\begin{tikzcd} G\times G\times G \arrow[r, "m\times id"] \arrow[d, "id\times m"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd} \qquad (1)$$

- the unit axiom, which is the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "\mathbb{1}\times id"] \arrow[d, "id\times\mathbb{1}"'] \arrow[dr, "id"] & G\times G \arrow[d, "m"] \\
G\times G \arrow[r, "m"'] & G
\end{tikzcd}$$

- the inverse axiom, which is the following comm've diagram

$$\begin{tikzcd}
G \arrow[rr, "(id, i)"] \arrow[dd, "(i, id)"'] \arrow[dr] & & G\times G \arrow[dd, "m"] \\
& pt \arrow[dr, "\mathbb{1}"] & \\
G\times G \arrow[rr, "m"'] & & G
\end{tikzcd}$$

Now consider the pullback homomorphisms

$$m^*: \mathbb{F}[G] \longrightarrow \mathbb{F}[G\times G] = \mathbb{F}[G]\otimes\mathbb{F}[G]$$
$$\mathbb{1}^*: \mathbb{F}[G] \longrightarrow \mathbb{F}[pt] = \mathbb{F}$$
$$i^*: \mathbb{F}[G] \longrightarrow \mathbb{F}[G]$$

The diagrams (1)-(3) translate to the corresponding diagrams for $m^*, \mathbb{1}^*, i^*$ – replace the varieties with their algebras of functions and reverse all arrows.

We arrive at the following definition

Definition (Hopf algebra)

Let $\mathbb{F}$ be a field and A be an associative unital

$\mathbb{F}$-algebra. We write $\mu: A\otimes_{\mathbb{F}} A \to A$ for the multiplication map: $\mu(a\otimes b) = ab$, and $\varepsilon: \mathbb{F} \to A$ for the unit map, $z \mapsto z1$.

By a Hopf algebra structure we mean a triple of algebra homomorphisms:

- The coproduct $\Delta: A \to A\otimes_{\mathbb{F}} A$
- The counit $\eta: A \to \mathbb{F}$
- The antipode $S: A \to A^{opp}$ (the algebra w. opposite product)

that should make the following diagrams commutative

$$\begin{tikzcd} A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id"'] \\ A\otimes A \arrow[u, "id\otimes\Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad (1')$$

$$\begin{tikzcd} A & A\otimes A \arrow[l, "\eta\otimes id"'] \\ A\otimes A \arrow[u, "id\otimes\eta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "id"'] \end{tikzcd} \qquad (2')$$

$$\begin{tikzcd} A & & A\otimes A \arrow[ll, "\mu\circ(id\otimes S)"'] \\ & \mathbb{F} \arrow[ul, "\varepsilon"'] & \\ A\otimes A \arrow[uu, "\mu\circ(S\otimes id)"] & & A \arrow[ll, "\Delta"] \arrow[uu, "\Delta"'] \arrow[ul, "\eta"'] \end{tikzcd} \qquad (3')$$

.

Example 1: $\mathbb{F}[G]$ is a Hopf algebra w. $\Delta = m^*$, $S = \iota^*$, $\eta = 1^*$.

Example 2: Let G be a group. Then the group algebra $\mathbb{F}G$ is a Hopf algebra w. $\Delta(g) = g\otimes g$, $\eta(g) = 1$, $S(g) = g^{-1}$. To check the axioms is an exercise.

There are further examples: the cohomology algebra of a Lie group (that was studied by Hopf), the universal enveloping algebra of a Lie algebra and, perhaps the most interesting the quantum groups, q-deformed versions of universal enveloping algebras.

Rem: The definition of a Hopf algebra is "self-dual." More precisely, let A be <u>a finite dimensional</u> Hopf algebra w. operations $\mu: A\otimes A \to A$, $\varepsilon: \mathbb{F}\to A$, $\Delta: A\to A\otimes A$, $\eta: A\to\mathbb{F}$, $S: A\to A^{opp}$. One can check that A^* is a Hopf algebra w.r.t. $(\Delta^*, \eta^*, \mu^*, \varepsilon^*, S^*)$, an exercise. For example, for a finite group G, we have $(\mathbb{F}G)^* = \mathbb{F}[G]$, the algebra of functions on G (also an exercise).

2) Distribution algebras.

2.1) Definition

The construction of the previous remark doesn't work well when $\dim A = \infty$, for example, in this case the inclusion $A^*\otimes A^* \subset$

$(A\otimes A)^*$ is proper. In this section we will explain a construction of the "correct" dual of the Hopf algebra $\mathbb{F}[G]$, where G is an algebraic group.

We start with defining the notion of a distribution. Let X be an affine variety and $\alpha\in X$. Let $\mathfrak{m}\subset\mathbb{F}[X]$ be the maximal ideal corresponding to α.

Definition: By a distribution at α we mean an element of $\mathbb{F}[X]^*$ that vanishes on $\mathfrak{m}^n$ for some $n>0$ (depending on the element). Let $Dist_\alpha$ denote the subspace of distributions in $\mathbb{F}[X]^*$, so that $Dist_\alpha=\bigcup_{n\geq 1}(\mathbb{F}[X]/\mathfrak{m}^n)^*$.

The space $Dist_\alpha$ doesn't have an algebra structure but has a natural coproduct, Δ, and counit η. The latter is given by evaluation at $1\in\mathbb{F}[X]$. To define Δ note that $\mathbb{F}[X]/\mathfrak{m}^n$ is an algebra & if μ_n denotes the product map:

$$\mu_n:\mathbb{F}[X]/\mathfrak{m}^n\otimes\mathbb{F}[X]/\mathfrak{m}^n\longrightarrow\mathbb{F}[X]/\mathfrak{m}^n$$

For $\delta\in(\mathbb{F}[X]/\mathfrak{m}^n)^*\subset Dist_\alpha$ set

$$\Delta(\delta)=\mu_n^*(\delta)\subset(\mathbb{F}[X]/\mathfrak{m}^n)^*\otimes(\mathbb{F}[X]/\mathfrak{m}^n)^*\subset Dist_\alpha\otimes Dist_\alpha$$

An easy check (exercise) shows that $\Delta(\delta)$ is independent of the choice of n.

When X is an algebraic group G & $\alpha=1$, the space $Dist_1(G)$ acquires an algebra structure. The remaining maps are as follows

• The product map $\mu: Dist_1(G)\otimes Dist_1(G) \longrightarrow Dist_1(G)$ comes from the product in the group: for $\delta_1, \delta_2 \in Dist_1(G), f\in \mathbb{F}[G]$, we have $[\mu(\delta_1\otimes\delta_2)](f) = [\delta_1\otimes\delta_2](m^* f)$, where $m^*: \mathbb{F}[G]\to \mathbb{F}[G]\otimes\mathbb{F}[G]$ is the pullback under the product $m: G\times G\to G$.

Exercise: μ is well-defined: $\mu(\delta_1\otimes\delta_2)$, a priori, an element of $\mathbb{F}[G]^*$, is, in fact, in $Dist_1(G)$.

• The unit map $\varepsilon: \mathbb{F}\to Dist_1(G)$ comes from the unit in the group, it sends $1\in\mathbb{F}$ to $f\mapsto f(1)$ $(1\in G)$

• The antipode $S: Dist_1(G)\longrightarrow Dist_1(G)$ is given by $\delta\mapsto \delta\circ i^*$, where $i: G\to G$ is the inversion map.

Exercise: Show that $Dist_1(G)$ with these operations is indeed a Hopf algebra.

2.2) Examples.

Example 1:

Consider the case of the additive group $G=\mathbb{F}$ (in this case 0 is the group unit). Let δ_i, $i\in\mathbb{Z}_{\geqslant 0}$, be defined by $\delta_i(x^j)=\delta_{ij}$. The elements δ_i form a vector space basis in $Dist_0(\mathbb{F})$. We will determine $\Delta(\delta_i)$ and $\delta_i\delta_j$ $(=\mu(\delta_i\otimes\delta_j))$. The other operations are left as an exercise.

$$(4)\begin{cases} \Delta(\delta_i)(x^k\otimes x^\ell) = \delta_i(x^{k+\ell}) = \delta_{i,k+\ell} \Rightarrow \Delta(\delta_i) = \sum_{j=0}^{i} \delta_j\otimes\delta_{i-j}. \\ \delta_i\delta_j(x^k) = [\text{the coefficient of } x^i y^j \text{ in } (x+y)^k] = \binom{i+j}{i}\delta_{i+j}. \end{cases}$$

Rem: Here's an alternative way to think about this Hopf algebra. Consider the Hopf $\mathbb{Q}$-algebra $\mathbb{Q}[\delta]$ w. $\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta$ (and $S(\delta) = -\delta$, $\eta(\delta) = 0$) - since Δ, S, η are algebra homomorphisms, they are determined by the image of a generator. Inside, consider $A := \mathrm{Span}_{\mathbb{Z}}\left(\frac{\delta^i}{i!} \mid i \geq 0\right)$. It's a subring and, actually, a Hopf subring meaning that $\Delta(A) \subset A \otimes_{\mathbb{Z}} A \hookrightarrow \mathbb{Q}[\delta] \otimes_{\mathbb{Q}} \mathbb{Q}[\delta]$ (and similar conditions for S & η). So $\mathbb{F} \otimes_{\mathbb{Z}} A$ has an induced Hopf algebra structure.

Exercise: Show that the $\mathbb{F}$-linear map $\mathbb{F} \otimes_{\mathbb{Z}} A \longrightarrow \mathrm{Dist}(\mathbb{G}_a)$ given by $1 \otimes \frac{\delta^i}{i!} \mapsto \delta_i$ is a Hopf algebra isomorphism.

Example 2: Consider the multiplicative group $\mathbb{F}^\times$. We compute the Hopf algebra $\mathrm{Dist}_1(\mathbb{F}^\times)$. As a vector space, $\mathrm{Dist}_1(\mathbb{F}^\times) = \bigcup_{n \geq 1} \left(\mathbb{F}[x^{\pm 1}]/(x-1)^n\right)^*$. Note that $\mathbb{F}[x]/(x-1)^n \xrightarrow{\sim} \mathbb{F}[x^{\pm 1}]/(x-1)^n$ so $\mathbb{F}[x^{\pm 1}]/(x-1)^n$ has basis of the cosets of $1, \ldots, (x-1)^{n-1}$. It follows that $\mathrm{Dist}_1(\mathbb{F}^\times)$ has basis $\delta_0, \delta_1, \ldots, \delta_n, \ldots$ given by $\delta_i((x-1)^j) = \delta_{ij}$. The coproduct $\Delta(\delta_i)$ is given by the same formula as in Ex 1. To compute the product note that $\Delta(x) = x \otimes x$ so setting $y = x-1$ we get $\Delta(y \otimes y) = y \otimes y + y \otimes 1 + 1 \otimes y$. Using some classical combinatorics we arrive at: for

$$\delta_n \delta_m = \sum_{i=0}^{\min(m,n)} \frac{(m+n-i)!}{(m-i)!(n-i)!\,i!}\, \delta_{n+m-i}.$$

Rem: This has a more transparent interpretation similar to the remark for Ex 1. Note that $\delta_1\delta_r = (r+1)\delta_r + r\delta_r \Rightarrow (\delta_1 - r)\delta_r = (r+1)\delta_{r+1}$ $\Rightarrow r!\,\delta_r = \delta_1(\delta_1 - 1)\ldots(\delta_1 - (r-1))$ $\forall\, r \geq 2$. Consider the Hopf $\mathbb{Q}$-algebra $\mathbb{Q}[\delta]$ as in Ex 1. Then consider the abelian subgroup B spanned by the elements $\binom{\delta}{r} = \frac{\delta(\delta-1)\ldots(\delta-(r-1))}{r!}$ for $r \geq 0$. It's again a Hopf subring. And $\mathbb{F} \otimes_{\mathbb{Z}} B \xrightarrow{\sim} \mathrm{Dist}_1(\mathbb{F}^\times)$