Representation theory of algebraic groups & Lie algebras, II.

0) Motivation

1) Tangent spaces in Algebraic geometry.

2) Bracket on the tangent space at 1 of an algebraic group.

0) Algebraic (or Lie) groups are non-linear objects (defined by non-linear equations). A basic paradigm to study such objects is linearization. In the context of Lie groups, this was applied already by Sophus Lie, leading to the notion of Lie algebras. If the base field has characteristic 0, the study of structure and representation theory of Lie or algebraic groups reduces to those for the Lie algebras. In characteristic p, the two are still related but the relation is more subtle.

1.1) Definitions

Let $\mathbb{F}$ be an algebraically closed field and X be an affine algebraic variety. We write $\mathbb{F}[X]$ for the algebra of polynomial functions on X. Pick $\alpha \in X$.

Definition: An α-derivation of $\mathbb{F}[X]$ is an $\mathbb{F}$-linear map $\delta: \mathbb{F}[X] \to \mathbb{F}$ satisfying the following version of Leibniz identity:

$$\delta(fg) = f(\alpha)\delta(g) + g(\alpha)\delta(f).$$

Note that the α-derivations form a vector subspace in the space $\mathbb{F}[X]^*$ of linear functions $\mathbb{F}[X] \to \mathbb{F}$. The space of α-derivations is denoted by $T_\alpha X$ and is called the tangent space of X at α.

Here is an explicit characterization of $T_\alpha X$. Suppose $\mathbb{F}[X] = \mathbb{F}[x_1,\dots x_n]/(f_1,\dots,f_m)$, in particular $X \hookrightarrow \mathbb{F}^n$. Then the pullback to $\mathbb{F}[x_1,\dots x_n]$ identifies $T_\alpha X$ w. $\{\delta \in T_\alpha \mathbb{F}^n \mid \delta(f_j)=0, j=1,\dots m\}$, exercise.

Since $\delta(f_j) = \sum_{i=1}^n (\partial_i f_j)(\alpha)\cdot \delta(x_i)$, the map $\delta \mapsto (\delta(x_1),\dots \delta(x_n))$ gives an isom'm $\{\delta \in T_\alpha \mathbb{F}^n \mid \delta(f_j)=0\} \xrightarrow{\sim} \{v \in \mathbb{F}^n \mid \sum_{i=1}^n v_i [\partial_i f_j](\alpha) = 0, j=1,\dots m\}$.

Now let Y be another affine variety, $\varphi: X \to Y$ be a morphism & $\beta = \varphi(\alpha)$. Consider the pullback homomorphism $\varphi^*: \mathbb{F}[Y] \to \mathbb{F}[X]$

Exercise : If δ is an α-derivation, then $\delta\circ\varphi^*$ is a β-derivation. The map $\delta \mapsto \delta\circ\varphi^*$ is a linear map $T_\alpha X \to T_\beta Y$.

Definition: This map is called the tangent map of φ at α and is denoted $T_\alpha \varphi$.

1.2) Examples:

We want to compute the spaces $T_1 G$ for the classical groups, Examples 0-3 in Sec 1.1 of Lec 5, where we write 1 for the identity matrix. Here's an easy case.

Example 0: If $U = X_f$ w. $f \in \mathbb{F}[X]$, then $T_\alpha U \xrightarrow{\sim} T_\alpha X$, $\forall \alpha \in U$, exercise.

For $GL_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})_{\det}$, get $T_1 GL_n(\mathbb{F}) \xrightarrow{\sim} T_1 \mathrm{Mat}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$. One commonly uses the notation $\mathfrak{gl}_n(\mathbb{F})$ for $\mathrm{Mat}_n(\mathbb{F})$ in this context.

To handle $G = SL_n, O_n, Sp_n$ we use two observations. First, if

X is an affine variety, and $Y \subset X$ is a Zariski closed subvariety, then, for the inclusion $i: Y \hookrightarrow X$ we have $T_\alpha i : T_\alpha Y \hookrightarrow T_\alpha X$. This is because $i^*: \mathbb{F}[X] \twoheadrightarrow \mathbb{F}[Y]$ and so $\delta \mapsto \delta \circ i^*$: $T_\alpha Y \to T_\alpha X$ is injective. So $T_1 SL_n, T_1 O_n, T_1 Sp_n \hookrightarrow \mathfrak{gl}_n(\mathbb{F})$.

Second, we will need the following version of the regular value theorem (w/o proof, see Sec 5.5 in [H2] for a related statement).

Fact: Let U be an affine variety that is open in some $\mathbb{F}^m$ & $\varphi: U \to \mathbb{F}^k$ be a morphism. Let $\alpha \in U$ be such that $T_\alpha \varphi : T_\alpha U \to T_{\varphi(\alpha)} \mathbb{F}^k$ is surjective. Then

$$T_\alpha[\varphi^{-1}(\varphi(\alpha))] = \ker T_\alpha \varphi.$$

We will apply this to $U = GL_n(\mathbb{F})$. In all cases we care about $G = \varphi^{-1}(\beta)$ for $\beta \in \mathbb{F}^k$ for suitable k.

Example 1: Let $\varphi: U \to \mathbb{F}$ be given by $g \mapsto \det(g)$. For $\xi \in T_1 U = \mathfrak{gl}_n(\mathbb{F})$, have $[T_1(\det)](\xi) = \frac{d}{ds} \det(1+\xi s)|_{s=0} = tr(\xi)$. So $T_1(\det) = tr$. The map $\xi \mapsto tr(\xi): \mathfrak{gl}_n(\mathbb{F}) \to \mathbb{F}$ is surjective. We conclude that $T_1(SL_n(\mathbb{F})) = T_1[\varphi^{-1}(1)]$ is $\{\xi \in \mathfrak{gl}_n(\mathbb{F}) \mid tr(\xi) = 0\} =: \mathfrak{sl}_n(\mathbb{F})$.

Example 2: Assume char $\mathbb{F} \neq 2$ and let $Y := \{\text{symmetric matrices}\} \subset \mathfrak{gl}_n(\mathbb{F})$. Take $\varphi: U \to Y$, $g \mapsto g^T g$. Then $[T_1 \varphi](\xi) = \xi^T + \xi$, similarly to Example 1. This map is surjective. Since $O_n(\mathbb{F}) = \varphi^{-1}(1)$, we get

$T_1 O_n(\mathbb{F}) = \ker T_1\Phi = \{\xi \in \mathfrak{gl}_n(\mathbb{F}) \mid \xi^T = -\xi\} =: \mathfrak{so}_n(\mathbb{F})$

We also could (and should) view this basis-free: $O(V,B)$ instead of $O_n(\mathbb{F})$. We get $T_1 O(V,B) = \{\xi \in \mathfrak{gl}(V) \mid B(\xi u, v) + B(u, \xi v) = 0 \ \forall u,v \in V\} =: \mathfrak{so}(V,B)$ – the space of operators skew-symmetric w.r.t. B.

Exercise: $T_1 Sp(V,\omega) = \{\xi \in \mathfrak{gl}(V) \mid \omega(\xi u, v) + \omega(u, \xi v) = 0 \ \forall u,v \in V\} =: \mathfrak{sp}(V,\omega)$.

2) Bracket on the tangent space at 1 of an algebraic group

For $\xi, \eta \in \mathfrak{gl}_n(\mathbb{F})$ we write $[\xi,\eta] := \xi\eta - \eta\xi$. For an algebraic group $G \subset GL_n(\mathbb{F})$ we write $\mathfrak{g}$ for $T_1 G$, as mentioned in Sec 1.2, this is a subspace in $\mathfrak{gl}_n(\mathbb{F})$.

Thm: 1) $\mathfrak{g}$ is closed under $[\cdot,\cdot]$.

2) Moreover, if H is another algebraic group, $\Phi: G \to H$ is an alg. group homomorphism, and $\mathfrak{h} := T_1 H$, $\varphi := T_1\Phi$, then $\varphi([\xi,\eta]) = [\varphi(\xi), \varphi(\eta)] \ \forall \xi,\eta \in \mathfrak{g}$.

Exercise: Check 1) explicitly for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}(V,B), \mathfrak{sp}(V,\omega)$.

Proof: Step 1: We produce a bilinear map $[\cdot,\cdot]': \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}[G]^*$. Recall that $\xi_1, \xi_2 \in \mathfrak{g}$ can be viewed as linear functions on $\mathbb{F}[G]$. So $\xi_1 \otimes \xi_2$ is a linear function $\mathbb{F}[G] \otimes \mathbb{F}[G] \to \mathbb{F}$.

Recall that $\mathbb{F}[G]\otimes\mathbb{F}[G] \xrightarrow{\sim} \mathbb{F}[G\times G]$. Consider the map $C(=C_G): G\times G \longrightarrow G$, $C(g_1,g_2)=g_1g_2g_1^{-1}g_2^{-1}$. Since the multiplication and inversion maps are given by polynomial functions, the same is true for C, i.e. C is a morphism. So we have the pullback homomorphism $C^*: \mathbb{F}[G]\to\mathbb{F}[G\times G]$. Set

$[\xi_1,\xi_2]' := [\xi\otimes\eta]\circ C^*: \mathbb{F}[G]\to\mathbb{F}$. It's indeed bilinear.

Step 2: We compute $[\xi_1,\xi_2]'$ for $G=GL_n(\mathbb{F})$. We have

$$[\xi_1,\xi_2]'(f) = \xi_1\otimes\xi_2 . f(g_1g_2g_1^{-1}g_2^{-1}) = [\xi_i \text{ differentiates w.r.t } g_i]$$

$$= \partial_{s_1}\partial_{s_2} f\Big(\underbrace{(1+s_1\xi_1)(1+s_2\xi_2)(1+s_1\xi_1)^{-1}(1+s_2\xi_2)^{-1}}_{=\left[\,1+s_1s_2([\xi_1,\xi_2]+O(s_1,s_2))\,\right]}\Big)\Big|_{s_1=s_2=0} =$$

exercise

$$= \partial_{s_1}\partial_{s_2} f(1+s_1s_2[\xi_1,\xi_2])\big|_{s_1=s_2=0} = T_1 f([\xi_1,\xi_2]),$$

where we view $T_1 f$ as an element in $\mathrm{Mat}_n(\mathbb{F})^*$. But then $T_1f([\xi_1,\xi_2]) = [\xi_1,\xi_2].f$ for $[\xi_1,\xi_2]$ viewed as an element of T_1G.

Step 3: In the notation of 2), we claim that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{g}\times\mathfrak{g} \arrow[r, "{[\cdot,\cdot]'}"] \arrow[d, "\varphi\times\varphi"'] & \mathbb{F}[G]^* \arrow[d, "?\circ\varphi^*"] \\
\mathfrak{h}\times\mathfrak{h} \arrow[r, "{[\cdot,\cdot]'}"] & \mathbb{F}[H]^*
\end{tikzcd} \qquad (*)$$

Indeed, recall that φ is a group homomomorphism, so
$$\varphi\circ c_G = c_H\circ(\varphi\times\varphi) \Leftrightarrow (\varphi\otimes\varphi)^*\circ c_H^* = c_G^*\circ\varphi^*.$$
Then $[\varphi(\xi_1),\varphi(\xi_2)]' = (\varphi(\xi_1)\otimes\varphi(\xi_2))\circ c_H^* = (\xi_1\otimes\xi_2)\circ(\varphi\times\varphi)^*\circ c_H^*$
$= (\xi_1\otimes\xi_2)\circ c_G^*\circ\varphi^* = [\xi_1,\xi_2]'\circ\varphi^*$. So $(*)$ is commutative.

Step 4: Apply $(*)$ to $H = GL_n$ & $\varphi :=$ the inclusion $i: G\hookrightarrow H$. Then $\mathbb{F}[H]\twoheadrightarrow\mathbb{F}[G]\Rightarrow\mathbb{F}[G]^*\hookrightarrow\mathbb{F}[H]^*$. For $\delta\in\mathbb{F}[G]^*$, the Leibniz identity $\delta(fg) = f(1)\delta(g)+g(1)\delta(f)$ is independent of whether we view δ as an element of $\mathbb{F}[H]^*$ or an element of $\mathbb{F}[G]^*$. So $\mathfrak{g} = \mathfrak{h}\cap\mathbb{F}[G]^*$.
The commutativity of $(*)$ implies $[\xi_1,\xi_2]'\in\mathfrak{g}$, and, thx to Step 2, $[\xi_1,\xi_2]' = [\xi_1,\xi_2]$. This implies Claim 1 of the theorem.

Step 5: Claim 2 follows from $(*)$ and the observation of Step 4 that $\operatorname{im}[\cdot,\cdot]'\subset T_1$ for any algebraic group. $\square$

Bonus remark: there are other equivalent definitions of the bracket on $\mathfrak{g}$ that we are going to sketch now.

1) For an affine variety X we can talk about vector fields on X. By definition, these are derivations $\mathbb{F}[X]\to\mathbb{F}[X]$, i.e. $\mathbb{F}$-linear maps ξ satisfying the Leibniz identity: $\xi(fg) = f\xi(g)+g\xi(f)$, $\forall f,g\in\mathbb{F}[X]$. Denote the space of derivations by $\mathrm{Vect}(X)$. It comes with a bracket: for $\xi,\eta\in\mathrm{Vect}(X)$, the map
$\xi\circ\eta-\eta\circ\xi:\mathbb{F}[X]\to\mathbb{F}[X]$ is a derivation.

Now let $X = G$ be an algebraic group. The group G acts on $\mathrm{Vect}(G)$ (say, via the action $G \curvearrowright G$ on the right) and the action respects the bracket. So the subspace $\mathrm{Vect}(G)^G$ of invariant vector fields is stable under the bracket. Similarly to the C^∞-case restricting a vector field to $1 \in G$ gives an isomorphism intertwining the brackets. The functoriality-part 2 of the theorem is then harder to establish, see [H], Sec 9.2.

2) One can adapt an approach from [OV], Sec 1.2, to the algebraic setting as follows. For $k > 0$, consider the algebra $\mathbb{F}[\varepsilon]/(\varepsilon^k)$ of "truncated polynomials". For an algebraic group G, we consider its "group of $\mathbb{F}[\varepsilon]/(\varepsilon^k)$-points". A naive definition is as follows: G is defined inside $GL_n(\mathbb{F})$ by some polynomial equations. Consider the subset of $GL_n(\mathbb{F}[\varepsilon]/(\varepsilon^k))$, the group of invertible matrices w. entries in $\mathbb{F}[\varepsilon]/(\varepsilon^k)$, given by the <u>same</u> polynomial equations. It's a subgroup. A more conceptual way is to view this group as the group of scheme morphisms $\mathrm{Spec}(\mathbb{F}[\varepsilon]/(\varepsilon^k)) \longrightarrow G$ (that should be viewed as "curves up to order k in G"). Denote the resulting group by G_k. Note that for $k < \ell$ we have a homomorphism of (abstract) groups $G_\ell \to G_k$. An exercise is to check that the kernels of $G_3 \to G_2$ and $G_2 \to G_1 = G$ are identified with $\mathfrak{g}$ (w. its additive group structure).

Now a group homomorphism $\Phi: G \to H$ gives rise to a group homomorphism $G_k \to H_k$ for all k (given by the same polynomials), this is especially easy to see if we identify G_k (resp. H_k) w. the group of morphisms from $\mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^k))$ to G (resp. H), then we just postcompose. For each $k<\ell$, the diagram

$$\begin{array}{ccc} G_\ell & \longrightarrow & H_\ell \\ \downarrow & & \downarrow \\ G_k & \longrightarrow & H_k \end{array} \qquad (*)$$

is commutative. The induced homomorphisms

$$\mathfrak{g} = \ker[G_2 \to G_1] \longrightarrow \ker[H_2 \to H_1] = \mathfrak{h}$$
$$\mathfrak{g} = \ker[G_3 \to G_2] \longrightarrow \ker[H_3 \to H_2] = \mathfrak{h}$$

coincide w. $T_1\varphi$. Note that the kernels above are abelian. So the commutator map for $\ker[G_3 \to G_1]$ descends to a map $\ker[G_2 \to G_1] \times \ker[G_2 \to G_1] \longrightarrow \ker[G_3 \to G_2]$. Under the identification of these kernels w. $\mathfrak{g}$, we recover the bracket $[\cdot,\cdot]$ on $\mathfrak{g}$. And from $(*)$ we deduce that $T_1\Phi$ intertwines the brackets.