1.1) Definitions & basic examples. Let $F$ be a field.

**Definition 1:** A Lie algebra over $F$ is an $F$-vector space $g$ equipped with a bilinear map $\langle \cdot , \cdot \rangle : g \times g \to g$ (Lie bracket or commutator), satisfying the following two properties:

- **Skew-symmetry:** $[x,x] = 0 \ \forall x \in g \ (\Rightarrow [g,y] = -[y,x] \ \forall y \in g$, exercise).
- **Jacobi identity:** $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

Once one knows the skew-symmetry, the Jacobi identity is equivalent to:

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]] \quad (1)$$

A Lie algebra homomorphism is an $F$-linear map $q : g \to k$ s.t. $q(x)q(y) = q([x,y])$.

**Example 0:** Abelian Lie algebra: $\langle \cdot , \cdot \rangle = 0$.

**Example 1:** Let $A$ be an associative algebra. Then $[a,b] := ab - ba$ is a Lie bracket (exercise). An important special case: $A = \text{Mat}_n(F)$ (or $\text{End}(V)$ for a vector space $V$). The resulting Lie algebra is denoted by $\mathfrak{gl}_n(F)$ (or $\mathfrak{gl}(V)$).
Example 2: Let $G \subset G_n(F)$ be an algebraic group. Then $g_1 : T_1 G$ is a Lie subalgebra in $g_f_1 (F)$ (1) in Thm from Sec 2 in Lec 6), so is a Lie algebra. Moreover, by (2) of that Thm, for an algebraic group homomorphism $\Phi : G \to H$, its tangent map $g_1 : T_1 \Phi$ is a Lie algebra homomorphism $g_1 \to g_1$.

1.2) Representations of Lie algebras.

As usual, a representation of a Lie algebra in a vector space $V$ is a Lie algebra homomorphism $g_1 \to g_l(V)$.

Example 1 (adjoint representation): For $x \in g_1$, let $\text{ad}(x) : g_1 \to g_1$ denote the operator $z \mapsto [x, z]$; $\text{ad} : g_1 \to g_f_1 (g_1)$ is a representation $\text{ad}([x, y]) = [\text{ad}(x), \text{ad}(y)]$ is (1) in Sec. 1.1 — called the adjoint representation. Moreover, if $G$ is an algebraic group and $g_1$ is its Lie algebra, then this representation arises as the tangent map of a rational representation of $G$ in $g_1$ (also called the adjoint representation). $\text{Ad} : G \to G(g_1)$. $G$ acts on itself by the adjoint action: $g \cdot (g') = g g' g^{-1}$. Since $a_0 (g) = 1$, $g \mapsto T_1 a_0$ gives a representation $\text{Ad} : G \to G(g_1)$. It is rational & $T_1 \text{Ad} = \text{ad}$. Indeed, first consider $G = G_n (F)$. Then $\text{Ad}(g) z = g z g^{-1}$ $(z \in g_1 = g_f_1 (F)) \Rightarrow T_1 \text{Ad}(x) z = [x, z]$ (exercise). For the general $G$, embed $G \subset G_n (F)$, and consider the representations $\tilde{\text{Ad}}$ of $G$ in $g_f_1 (F)$: $\tilde{\text{Ad}}(g) z = g z g^{-1}$ & $\tilde{\text{ad}}$ of $g_1$ in $g_f_1 (F)$ so that $T_1 \tilde{\text{Ad}} = \tilde{\text{ad}}$.

Exercise: $g_1 \subset g_f_1 (F)$ is a subrepresentation for both, the resulting
representations of $G$ & $g$ in $g$ coincide with Ad & $\text{ad}$. So Ad is rational & $T,A\text{d}=\text{ad}$.

**Example 2:** Let $V$, $W$ be representations of a Lie algebra $g$. Then $V \otimes W$ has a unique structure of a representation of $g$ s.t.

$$x(v \otimes w) = xv \otimes w + v \otimes wx, \forall v \in V, w \in W$$ (2)

**Exercise:** 1) Check this is indeed a representation (tensor product rep'n).

2) Suppose that $V$, $W$ are rational representations of an algebraic group $G$, $g = T$, $G$ & representations of $g$ in $V$, $W$ are obtained by differentiating the representations of $G$. Then the representation of $g$ in $V \otimes W$ is obtained by differentiating the representation of $G$. This serves as a motivation for (2).

**Example 3:** If $V$ is a representation of $g$, then so is $V^*$ via

$$[x^*]_v = -x(v), \forall v \in V, x \in g$$ exercise. The motivation is similar to Example 2. This is the dual representation.

**Example 4:** If $F$ is a representation of $g$ where all $x \in g$ act by 0. This is the trivial representation.

1.3) Correspondence between algebraic groups & Lie algebras

**Fact:** Let $G$ be an affine algebraic group TFAE:

- $G$ is connected in the Zariski topology.
- $G$ is irreducible (as a variety)
Reason: $G$ is smooth as a variety. In this case we say $G$ is connected.

Example: $G = GL_n(F), SL_n(F), SO_n(F), Sp_n(F)$ are connected, $O_n(F)$ isn’t.

The irreducibility of $GL_n(F)$ is standard, for $SL_n(F)$ it follows from $\det -1$ being an irreducible polynomial (not so pleasant check). A general method is explained in the complement section.

**Theorem 1:** Suppose $G$ is connected and $\text{char } F = 0$. Let $H$ be another algebraic group, $\Phi_1, \Phi_2 : G \to H$ be algebraic group homomorphisms, $\Phi_i = T_1 \Phi_i$. If $\Phi_1 = \Phi_2$, then $\Phi_1 = \Phi_2$.

**Theorem 2:** Let $V, W$ be rational reps of $G$. If $\varphi : V \to W$ is $G$-linear, then it’s $g$-linear (exercise). If $G$ is connected, and $\text{char } F = 0$, then the converse is true as well.

**Remark:** Both Thms are false when $\text{char } F = p > 0$. For Thm 1, consider $G = H = GL_n(F), \Phi_1(g) = 1, \Phi_2 = F_r : (a_{ij}) \mapsto (a_{ij}^p)$. We have $\Phi_1 = \Phi_2 = 0$, but $\Phi_1 \neq \Phi_2$. A counterexample to Thm 2 will be provided later.

Sketch of proofs for $F = C$: $g_j = T_j G$ is identified with the Lie algebra $\text{Vect}(G)^G$ of left-invariant vector fields on $G$; for $g \in G$, let $\xi \in \text{Vect}(G)^G$ be the corresponding vector field. We write $\exp(t \xi)$ for the (parameterized) integral curve for $\xi$ through 1, it exists for all $t
(due to the invariance). For $G = GL_n(C), \exp(t\xi)$ is a solution of the differential equation $\frac{d}{dt} F(t) = F(t)\xi$, i.e. the usual matrix exponential: 
\[ \exp(t\xi) = \sum_{i=0}^{\infty} \frac{t^i}{i!}\xi^i. \]

Consider the map $\exp: \mathfrak{g} \rightarrow G$. It's a (complex) differentiable map sending 0 to 1, with tangent map at 0 being $\text{id}: \mathfrak{g} \rightarrow \mathfrak{g}$. Hence a neighborhood of 1 in $G$ lies in $\text{im}(\exp)$. So the subgroup in $G$ generated by $\exp(\mathfrak{g})$ is the connected component $G^0$ of 1 in the usual topology.

**Fact**: A variety over $C$ is connected in the usual topology iff it's connected in the Zariski topology (Hartshorne, Appendix B).

In particular, $\exp(\mathfrak{g})$ generates $G$.

Now let $\Phi: G \rightarrow H$ be a complex Lie group homomorphism. One can show that $\Phi(\exp(\xi)) = (\Phi \circ \exp)(\xi)$, $\forall \xi \in \mathfrak{g}$. It follows that $\Phi$ sends the integral curves for $\xi$ to integral curves for $\Phi(\xi)$. So,
\[ \Phi(\exp(\xi)) = \exp(\Phi(\xi)), \forall \xi \in \mathfrak{g}. \] (3)

**Theorem 1** follows. To prove **Thm 2** we write $g, \xi, \xi_v$ for the operators on $V$ corresponding to $g \in G, \xi \in \mathfrak{g}$. Then (3) applied to the homomorphism $g \rightarrow g_v: G \rightarrow G(V)$ implies $\Phi(\exp(\xi)) = \exp(\Phi(\xi))$.

Same for $W$. **Theorem 2** follows from here.

2) Universal enveloping algebra.

2.1) Definition.

The universal enveloping algebra for a Lie algebra $\mathfrak{g}$ plays the same role for Lie algebras as the group algebra for groups.
**Definition**: Define $U(g) = \frac{T(g)}{(x@y - y@x - [x,y])}$, where $T(g)$ is the tensor algebra of $g$.

The composition $g \hookrightarrow T(g) \rightarrow U(g)$ is a Lie algebra homomorphism. Here is the universal property of $U(g)$ (and this homomorphism).

**Lemma**: Let $A$ be an associative algebra (hence a Lie algebra, Ex 1 in Sec 1.1) and let $q: g \rightarrow A$ be a Lie algebra homomorphism. Then there is a unique associative algebra homomorphism $\hat{g}: U(g) \rightarrow A$ making the following diagram commutative:

$$
\begin{array}{ccc}
g & \rightarrow & \hat{g} \\
\downarrow & & \downarrow \\
U(g) & \rightarrow & A \\
\end{array}
$$

**Proof**: Since $q$ is an $F$-linear map, $\exists!$ assoc. algebra homomorphism $\hat{q}: T(g) \rightarrow A$ s.t. $g \hookrightarrow T(g) \hat{q} \rightarrow A$ coincides w. $q$. The condition that $q$ is a Lie algebra homomorphism means that $\hat{q}(x@y - y@x - [x,y]) = [\hat{q}(x), \hat{q}(y)] - \hat{q}([x,y]) = 0$ so $\hat{q}$ (uniquely) factors through the quotient $U(g)$ of $T(g)$). This gives the required $\hat{q}$. □

In particular, as for the groups vs group algebras, a representation of $g$ is the same thing as a $U(g)$-module.

**Example**: if $g$ is abelian, then $U(g) = \frac{T(g)}{(x@y - y@x)} = S(g) (\cong \mathbb{F}[g^{*}])$, the symmetric algebra of $g$. 
2.2) Poincaré-Birkhoff-Witt (PBW) theorem

Our goal is to establish a basis in $U(g)$. Assume for simplicity that $\dim g < \infty$. Let $x_1, \ldots, x_n$ be a basis in $g$. We can view any non-commutative polynomial in these elements as an element of $U(g)$.

Thm: The ordered monomials $x_1^{d_1} \cdots x_n^{d_n}$ form a basis in $U(g)$.

An easy part is that these elements span. A more precise claim is true. For $d \geq 0$, let $U(g)_{\leq d}$ denote the span of all monomials in $x_1, \ldots, x_n$ of degree $\leq d$.

Lemma: The ordered monomials $x_1^{d_1} \cdots x_n^{d_n}$ with $d_1 + \cdots + d_n \leq d$ span $U(g)_{\leq d}$.

Proof: exercise - induction on $d$ + observation that for $i < j$ have $x_i x_j = x_j x_i + [x_j, x_i]$, the 2nd summand is a linear combination of $x_i$'s.

The linear independence is more subtle, see [B], Ch. I, Sec 2.7 or [H1], Sec 17.4. The idea is to construct a representation of $g$ with basis $x_1^{d_1} \cdots x_n^{d_n}$ and the action given by left multiplication (where one needs to write the product $x_i x_j x_k^{d_1} x_1^{d_2} \cdots x_n^{d_n}$ as the linear combination of ordered monomials using $x_j x_i = x_i x_j + [x_j, x_i]$ with $j > i$.

The existence of such representation is automatic once we know the theorem - this is just $U(g)$ - but the point is it can be verified independently, although the check is unpleasant.
Complements

1) Checking connectedness and fundamental group

Example in Section 1.3 mentions that the groups $SL_n(F)$, $Sp_n(F)$, $SO_n(F)$ are connected. In this part we explain how to check this. This is done using the following observation:

Let $V$ be a representation of $G$. Suppose $v \in V$ is such that the orbit $Gv$ and the stabilizer $G_v$ are irreducible. Then $G$ is irreducible (equiv connected).

This can be applied as follows. Consider the case of $SL_n$. Take the tautological representation $F^n$ and take $v = (1, 0, \ldots, 0)$. The stabilizer $G_v$ is of the form \[
\begin{bmatrix}
1 & b \\
0 & A
\end{bmatrix}, a \in F^*, \det(A) = 1, b \in F^{n-1},
\]

$\cong F^{n-1} \times SL_{n-1}(F)$, an irreducible variety, by induction. The orbit $Gv$ is $F^n \setminus \{0\}$, also irreducible.

For $G = Sp_n$ we also consider its tautological representation and any nonzero vector. For $G = SO_n$, we consider the tautological representation and any vector $v$, nonzero square. The details are left as an exercise.

When $F = C$, a somewhat similar argument can be used to compute the fundamental group of $G$. Namely for any complex Lie subgroup $H$ we have an exact sequence (see [OV], Ch.4, Sec.3.4)

$\pi_1(H) \rightarrow \pi_1(G) \rightarrow \pi_1(G/H) \rightarrow \pi_0(G) \rightarrow \pi_0(H)$ \quad (*)$

$\pi_0(H)$: the group of connected components of $H$. This exact sequence $w. H = G_v$ allows to prove $\pi_1(SL_n(C)) = \{1\}$ if $n > 1$: $G/G_v = G_v$
= C^n \setminus \{0\}. For n > 1, this space is homotopic to S^{2n-1} hence is simply connected. So SL_n(C) is simply connected.

For similar reasons, Sp_n(C) is simply connected. On the other hand, \( \pi_1(SO_n(C)) \cong \mathbb{Z}/2\mathbb{Z} \) for \( n \geq 3 \). This is proved by induction, where the induction step is \( n \), while the base, \( n = 3 \), is handled using an isomorphism \( SL_2(C)/\{\pm I\} \cong SO_3(C) \) (proved using the action of \( SL_2(C) \) in its adjoint representation).

2) Existence results (for Lie/algebraic groups & homomorphisms)

2.1) Lie groups.

Here we consider the real Lie groups. The results easily carry over to complex Lie groups. Here's the main result.

Thm: 1) Every finite dimensional Lie algebra is the Lie algebra of a real Lie group.

2) This Lie group can be chosen to be simply connected.

3) Let G, H be connected Lie groups, \( g, h \) their Lie algebras & \( \varphi: g \to h \) be a Lie algebra homomorphism. If G is simply connected, then \( \exists \) a Lie group homomorphism \( \varphi: G \to H \) with \( \tilde{\varphi} \cdot \varphi = \varphi \).

3) is a technical statement proved for example in [OV], Ch. 1, Sec. 2.8, or [K], Sec. 3.8. To prove 2) one observes that the simply connected cover \( \tilde{G} \) of a Lie group G has a natural Lie group structure (and, moreover, \( G \cong \tilde{G}/\overline{Z} \), where \( \overline{Z} \) is a discrete central subgroup), see, e.g. [OV], Ch. 1, Sec 3.2.
1) is the most complicated: one

- either uses the Ado theorem that every finite dimensional Lie algebra is isomorphic to a subalgebra in some $gl_n(\mathbb{R})$
- or establishes the existence for semidirect products of Lie algebras and for semisimple Lie algebras, then uses the Levi theorem that every finite dimensional Lie algebra over $\mathbb{R}$ is isomorphic to the semidirect product of a semisimple & a solvable Lie algebra (and every solvable Lie algebra is realized as an iterated semidirect product of one-dimensional Lie algebras. This is the approach taken in [OV].

2.2) Algebraic groups over $\mathbb{C}$.

The situation with algebraic groups over $\mathbb{C}$ is more complicated. Details for this section can be found in [OV], Ch.3, Sec.3.

First, not every Lie algebra can be the Lie algebra of an algebraic group. Here are basic examples.

- take the subalgebra $\left\{(\begin{array}{ccc} \sqrt{a} & 0 & b \\ 0 & a & c \\ 0 & 0 & 0 \end{array}), a,b,c \in \mathbb{C} \right\} \subset gl_3(\mathbb{C})$.

- or the subalgebra $\left\{(\begin{array}{ccc} a & 0 & b \\ 0 & a & c \\ 0 & 0 & 0 \end{array}), a,b,c \in \mathbb{C} \right\} \subset gl_3(\mathbb{C})$.

Neither of these can be the Lie algebra of an algebraic group.

Next, part 3 of Thm in Sec.2.1 above also fails. Namely consider the 1-dimensional Lie algebra $\mathbb{C}$. It
corresponds to two algebraic groups: the additive group, $G_a$, and the multiplicative group, $G_m$. The former is simply connected, and there’s a surjective Lie group homomorphism $G_a \to G_m: z \mapsto \exp(z)$. It’s not algebraic and, in fact, there are no non-constant variety morphisms $C \to C^\times$.

The algebraic groups $G_a$ & $G_m$ behave very differently.

The situation is better for semisimple Lie algebras/algebraic groups (over $\mathbb{C}$ or, more generally, over $\mathbb{F}$ with char $\mathbb{F} = 0$). And over positive characteristic fields, the relationship is yet more complicated. We’ll address this in subsequent notes.