Representation theory of algebraic groups & Lie algebras, Part III.

1) Lie algebras

2) Universal enveloping algebras.

Complements.

1.1) Definitions & basic examples.

Let $\mathbb{F}$ be a field.

Definition 1: • A Lie algebra over $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ equipped w. a bilinear map $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (Lie bracket or commutator). satisfying the following two properties:

- Skew-symmetry: $[x,x]=0\ \forall x\in\mathfrak{g}$ ($\Rightarrow [x,y]=-[y,x]\ \forall x,y\in\mathfrak{g}$, exercise)
- Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$.

Once one knows the skew-symmetry the Jacobi id'y is equivalent to:

$$[[x,y],z]=[x,[y,z]]-[y,[x,z]] \qquad (1)$$

• A Lie algebra homomorphism is an $\mathbb{F}$-linear map $\varphi:\mathfrak{g}\to\mathfrak{h}$ s.t. $[\varphi(x),\varphi(y)]=\varphi([x,y])$

Example 0: Abelian Lie algebra: $[\cdot,\cdot]=0$.

Example 1: Let A be an associative algebra. Then $[a,b]:=ab-ba$ is a Lie bracket (exercise). An important special case: $A=\mathrm{Mat}_n(\mathbb{F})$ (or $\mathrm{End}(V)$ for a vector space V). The resulting Lie algebra is denoted by $\mathfrak{gl}_n(\mathbb{F})$ (or $\mathfrak{gl}(V)$).

Example 2: Let $G \subset GL_n(\mathbb{F})$ be an algebraic group. Then $\mathfrak{g} := T_1 G$ is a Lie subalgebra in $\mathfrak{gl}_n(\mathbb{F})$ ((1) in Thm from Sec 2 in Lec 6), so is a Lie algebra. Moreover, by (2) of that Thm, for an algebraic group homomorphism $\Phi: G \to H$, its tangent map $\varphi := T_1\Phi$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$.

1.2) Representations of Lie algebras.

As usual, a representation of a Lie algebra in a vector space V is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$.

Example 1 (adjoint representation): for $x \in \mathfrak{g}$, let $ad(x): \mathfrak{g} \to \mathfrak{g}$ denote the operator $z \mapsto [x,z]$; $ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation — $ad([x,y]) = [ad(x), ad(y)]$ is (1) in Sec. 1.1 — called the adjoint representation. Moreover, if G is an algebraic group and $\mathfrak{g}$ is its Lie algebra, then this representation arises as the tangent map of a rational representation of G in $\mathfrak{g}$ (also called the adjoint representation), $Ad: G \to GL(\mathfrak{g})$. G acts on itself by the adjoint action: $a_g(g') = gg'g^{-1}$. Since $a_g(1) = 1$, $g \mapsto T_1 a_g$ gives a representation $Ad: G \to GL(\mathfrak{g})$. It's rational & $T_1 Ad = ad$. Indeed, first consider $G = GL_n(\mathbb{F})$. Then $Ad(g)z = gzg^{-1}$ $(z \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})) \Rightarrow T_1 Ad(x)z = [x,z]$ (exercise). For the general G, embed $G \hookrightarrow GL_n(\mathbb{F})$, and consider the representations $\widetilde{Ad}$ of G in $\mathfrak{gl}_n(\mathbb{F})$: $\widetilde{Ad}(g)z = gzg^{-1}$ & $\widetilde{ad}$ of $\mathfrak{g}$ in $\mathfrak{gl}_n(\mathbb{F})$ so that $T_1\widetilde{Ad} = \widetilde{ad}$.

Exercise: $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{F})$ is a subrepresentation for both, the resulting representations of G & $\mathfrak{g}$ in $\mathfrak{g}$ coincide with Ad & ad. So Ad is rational & $T_1\mathrm{Ad} = \mathrm{ad}$.

Example 2: Let V, W be representations of a Lie algebra $\mathfrak{g}$. Then $V\otimes W$ has a unique structure of a representation of $\mathfrak{g}$ s.t.

$$x(v\otimes w) = xv\otimes w + v\otimes xw, \ \forall v\in V, w\in W. \qquad (2)$$

Exercise: 1) Check this is indeed a representation (tensor product rep'n)

2) Suppose that V, W are rational representations of an algebraic group G, $\mathfrak{g} = T_1 G$ & representations of $\mathfrak{g}$ in V, W are obtained by differentiating the representations of G. Then the representation of $\mathfrak{g}$ in $V\otimes W$ is obtained by differentiating the representation of G. This serves as a motivation for (2).

Example 3: If V is a representation of $\mathfrak{g}$, then so is V^* via $[x\eta](v) = -\eta(xv)$ $(v\in V, \eta\in V^*, x\in\mathfrak{g})$ – exercise. The motivation is similar to Example 2. This is the dual representation.

Example 4: $\mathbb{F}$ is a representation of $\mathfrak{g}$ where all $x\in\mathfrak{g}$ act by 0. This is the trivial representation.

1.3) Correspondence between algebraic groups & Lie algebras.

Fact: Let G be an affine algebraic group. TFAE:

- G is connected in the Zariski topology.
- G is irreducible (as a variety)

Reason: G is smooth as a variety. In this case we say G is connected.

Example: $G = GL_n(\mathbb{F}), SL_n(\mathbb{F}), SO_n(\mathbb{F}), Sp_n(\mathbb{F})$ are connected, $O_n(\mathbb{F})$ isn't.

The irreducibility of $GL_n(\mathbb{F})$ is standard, for $SL_n(\mathbb{F})$ it follows from $\det - 1$ being an irreducible polynomial (not so pleasant check). A general method is explained in the complement section.

Theorem 1: Suppose G is connected and $\operatorname{char} \mathbb{F} = 0$. Let H be another algebraic group, $\Phi_1, \Phi_2 : G \to H$ be algebraic group homomorphisms, $\varphi_i = T_1 \Phi_i$. If $\varphi_1 = \varphi_2$, then $\Phi_1 = \Phi_2$.

Theorem 2: Let V, W be rational repr's of G. If $\varphi : V \to W$ is G-linear, then it's $\mathfrak{g}$-linear (exercise). If G is connected, and $\operatorname{char} \mathbb{F} = 0$, then the converse is true as well.

Remark: Both Thm's are false when $\operatorname{char} \mathbb{F} = p > 0$. For Thm 1, consider $G = H = GL_n(\mathbb{F})$, $\Phi_1(g) = 1$, $\Phi_2 = Fr : (a_{ij}) \mapsto (a_{ij}^p)$. We have $\varphi_1 = \varphi_2 = 0$, but $\Phi_1 \neq \Phi_2$. A counterexample to Thm 2 will be provided later.

Sketch of proofs for $\mathbb{F} = \mathbb{C}$: $\mathfrak{g} = T_1 G$ is identified with the Lie algebra $\operatorname{Vect}(G)^G$ of left-invariant vector fields on G; for $\xi \in \mathfrak{g}$, let $\tilde{\xi} \in \operatorname{Vect}(G)^G$ be the corresponding vector field. We write $\exp(t\xi)$ for the (parameterized) integral curve for $\tilde{\xi}$ through 1, it exists for all t

(due to the invariance). For $G = GL_n(\mathbb{C})$, $\exp(t\xi)$ is a solution of the differential equation $\frac{d}{dt}F(t) = F(t)\xi$, i.e. the usual matrix exponential:

$$\exp(t\xi) = \sum_{i=0}^{\infty} \frac{t^i}{i!}\xi^i.$$

Consider the map $\exp: \mathfrak{g} \to G$. It's a (complex) differentiable map sending 0 to 1, with tangent map at 0 being $id: \mathfrak{g} \to \mathfrak{g}$. Hence a neigh'd of 1 in G lies in $im(\exp)$. So the subgroup in G generated by $\exp(\mathfrak{g})$ is the connected component G° of 1 in the usual topology.

Fact: A variety over $\mathbb{C}$ is connected in the usual topology iff it's connected in the Zariski topology (Hartschorne, Appendix B).

In particular, $\exp(\mathfrak{g})$ generates G.

Now let $\Phi: G \to H$ be a complex Lie group homomorphism. One can show that $\widetilde{\varphi(\xi)}_{\Phi(g)} = (T_g\Phi)(\tilde{\xi}_g)$, $\forall \xi \in \mathfrak{g}$. It follows that Φ sends the integral curves for $\tilde{\xi}$ to integral curves for $\widetilde{\varphi(\xi)}$. So,

$$\Phi(\exp \xi) = \exp \varphi(\xi), \ \forall \xi \in \mathfrak{g}. \qquad (3)$$

Theorem 1 follows. To prove Thm 2 we write g_V, ξ_V for the operators on V corresponding to $g \in G$, $\xi \in \mathfrak{g}$. Then (3) applied to the homomorphism $g \mapsto g_V: G \to GL(V)$ implies $(\exp(\xi))_V = \exp(\xi_V)$. Same for W. Theorem 2 follows from here.

2) Universal enveloping algebra.

2.1) Definition.

The universal enveloping algebra for a Lie algebra $\mathfrak{g}$ plays the same role for Lie algebras as the group algebra for groups.

Definition: Define $\mathcal{U}(\mathfrak{g}) := \dfrac{T(\mathfrak{g})}{(x\otimes y - y\otimes x - [x,y] \mid x,y\in\mathfrak{g})}$, where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$.

The composition $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow \mathcal{U}(\mathfrak{g})$ is a Lie algebra homomorphism. Here is the universal property of $\mathcal{U}(\mathfrak{g})$ (and this homomorphism).

Lemma: Let A be an associative algebra (hence a Lie algebra, Ex 1 in Sec 1.1) and let $\varphi: \mathfrak{g} \to A$ be a Lie algebra homomorphism. Then there is a unique associative algebra homomorphism $\tilde{\varphi}: \mathcal{U}(\mathfrak{g}) \to A$ making the following diagram commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[d] \arrow[dr, "\varphi"] & \\ \mathcal{U}(\mathfrak{g}) \arrow[r, dashed, "\tilde{\varphi}"'] & A \end{tikzcd}$$

Proof: Since φ is an $\mathbb{F}$-linear map, $\exists!$ assoc. algebra homomorphism $\hat{\varphi}: T(\mathfrak{g}) \to A$ s.t. $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \xrightarrow{\hat{\varphi}} A$ coincides w. φ. The condition that φ is a Lie algebra homomorphism means that $\hat{\varphi}(x\otimes y - y\otimes x - [x,y]) = [\varphi(x), \varphi(y)] - \varphi([x,y]) = 0$ so $\hat{\varphi}$ (uniquely) factors through the quotient $\mathcal{U}(\mathfrak{g})$ of $T(\mathfrak{g})$). This gives the required $\tilde{\varphi}$. $\square$

In particular, as for the groups vs group algebras, a representation of $\mathfrak{g}$ is the same thing as a $\mathcal{U}(\mathfrak{g})$-module.

Example: if $\mathfrak{g}$ is abelian, then $\mathcal{U}(\mathfrak{g}) = \dfrac{T(\mathfrak{g})}{(x\otimes y - y\otimes x)} = S(\mathfrak{g}) (\simeq \mathbb{F}[\mathfrak{g}^*])$, the symmetric algebra of $\mathfrak{g}$.

2.2) Poincare-Birkhoff-Witt (PBW) theorem

Our goal is to establish a basis in $\mathcal{U}(\mathfrak{g})$. Assume for simplicity that $\dim \mathfrak{g} < \infty$. Let $x_1, \dots x_n$ be a basis in $\mathfrak{g}$. We can view any non-commutative polynomial in these elements as an element of $\mathcal{U}(\mathfrak{g})$.

Thm: The <u>ordered</u> monomials $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ form a basis in $\mathcal{U}(\mathfrak{g})$.

An easy part is that these elements span. A more precise claim is true. For $d \geqslant 0$, let $\mathcal{U}(\mathfrak{g})_{\leqslant d}$ denote the span of all monomials in $x_1, \dots x_n$ of degree $\leqslant d$.

Lemma: The ordered monomials $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ w. $\alpha_1 + \dots + \alpha_n \leqslant d$ span $\mathcal{U}(\mathfrak{g})_{\leqslant d}$.

Proof: exercise – induction on d + observation that for $i<j$ have $x_j x_i = x_i x_j + [x_j, x_i]$, the 2nd summand is a linear combination of x_i's.

The linear independence is more subtle, see [B], Ch. I, Sec 2.7 or [H1], Sec 17.4. The idea is to construct a representation of $\mathfrak{g}$ w. basis $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and the action given by left multiplication (where one needs to write the product $x_\ell x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ as the linear combination of ordered monomials using $x_j x_i = x_i x_j + [x_j, x_i]$ w. $j>i$. The existence of such representation is automatic once we know the theorem – this is just $\mathcal{U}(\mathfrak{g})$ – but the point is it can be verified independly, although the check is unpleasant.

Complements

1) Checking connectedness and fundamental group.

Example in Section 1.3 mentions that the groups $SL_n(\mathbb{F})$, $Sp_n(\mathbb{F})$, $SO_n(\mathbb{F})$ are connected. In this part we explain how to check this. This is done using the following observation:

Let V be a representation of G. Suppose $v \in V$ is such that the orbit Gv and the stabilizer G_v are irreducible. Then G is irreducible (equiv. connected).

This can be applied as follows. Consider the case of SL_n. Take the tautological representation $\mathbb{F}^n$ and take $v = (1, 0, \dots, 0)$. The stabilizer G_v is of the form $\left\{ \begin{pmatrix} 1 & b \\ 0 & \\ 0 & A \\ \vdots & \\ 0 & \end{pmatrix} \middle| \, a \in \mathbb{F}^\times, \det(A) = 1, b \in \mathbb{F}^{n-1} \right\}$

$\simeq \mathbb{F}^{n-1} \times SL_{n-1}(\mathbb{F})$, an irreducible variety, by induction. The orbit Gv is $\mathbb{F}^n \setminus \{0\}$, also irreducible.

For $G = Sp_n$ we also consider its tautological representation and any nonzero vector. For $G = SO_n$, we consider the tautological representation and any vector w. nonzero square. The details are left as an exercise.

When $\mathbb{F} = \mathbb{C}$, a somewhat similar argument can be used to compute the fundamental group of G. Namely for any complex Lie subgroup H we have an exact sequence (see [OV], Ch 1, Sec 3.4)

$$\pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/H) \longrightarrow \pi_0(H) \qquad (*)$$

w. $\pi_0(H) :=$ the group of connected components of H. This exact sequence w. $H = G_v$ allows to prove $\pi_1(SL_n(\mathbb{C})) = \{1\}$ $\forall n > 1$: $G/G_v = Gv$

$=\mathbb{C}^n\setminus\{0\}$. For $n>1$, this space is homotopic to S^{2n-1} hence is simply connected. So $SL_n(\mathbb{C})$ is simply connected.

For similar reasons, $Sp_n(\mathbb{C})$ is simply connected. On the other hand, $\pi_1(SO_n(\mathbb{C}))\simeq \mathbb{Z}/2\mathbb{Z}$ for $n\geqslant 3$. This is proved by induction, where the induction step is (*), while the base, $n=3$, is handled using an isomorphism $SL_2(\mathbb{C})/\{\pm I\}\xrightarrow{\sim} SO_3(\mathbb{C})$ (proved using the action of $SL_2(\mathbb{C})$ in its adjoint representation).

2) Existence results (for Lie/algebraic groups & homomorphisms)

2.1) Lie groups.

Here we consider the real Lie groups. The results easily carry over to complex Lie groups. Here's the main result

Thm: 1) Every finite dimensional Lie algebra is the Lie algebra of a real Lie group.

2) This Lie group can be chosen to be simply connected.

3) Let G,H be connected Lie groups, $\mathfrak{g},\mathfrak{h}$ their Lie algebras & $\varphi:\mathfrak{g}\to\mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, then $\exists$ a Lie group homomorphism $\Phi: G\to H$ w. $T_1\Phi=\varphi$.

3) is a technical statement proved for example in [OV], Ch. 1, Sec 2.8, or [K], Sec 3.8. To prove 2) one observes that the simply connected cover $\tilde{G}$ of a Lie group G has a natural Lie group structure (and, moreover, $G\simeq \tilde{G}/Z$, where Z is a discrete central subgroup), see, e.g. [OV], Ch. 1, Sec 3.2.

1) is the most complicated: one

- either uses the Ado theorem that every finite dimensional Lie algebra is isomorphic to a subalgebra in some $\mathfrak{gl}_n(\mathbb{R})$.
- or establishes the existence for semidirect products of Lie algebras and for semisimple Lie algebras, then uses the Levi theorem that every finite dimensional Lie algebra over $\mathbb{R}$ is isomorphic to the semidirect product of a semisimple & a solvable Lie algebra (and every solvable Lie algebra is realized as an iterated semidirect product of one-dimensional Lie algebras. This is the approach taken in [OV].

2.2) Algebraic groups over $\mathbb{C}$.

The situation with algebraic groups over $\mathbb{C}$ is more complicated. Details for this section can be found in [OV], Ch 3, Sec 3.

First, not every Lie algebra can be the Lie algebra of an algebraic group. Here are basic examples.

take the subalgebra $\left\{\begin{pmatrix} \sqrt{2}a & 0 & b \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix}, a,b,c\in\mathbb{C}\right\}\subset\mathfrak{gl}_3(\mathbb{C})$.

or the subalgebra $\left\{\begin{pmatrix} a & a & b \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix}, a,b,c\in\mathbb{C}\right\}\subset\mathfrak{gl}_3(\mathbb{C})$.

Neither of these can be the Lie algebra of an algebraic group.

Next, part 3 of Thm in Sec 2.1 above also fails. Namely consider the 1-dimensional Lie algebra $\mathbb{C}$. It corresponds to two algebraic groups: the additive group, $\mathbb{G}_a$, and the multiplicative group, $\mathbb{G}_m$. The former is simply connected, and there's a surjective Lie group homomorphism $\mathbb{G}_a \to \mathbb{G}_m : z \mapsto \exp(z)$. It's not algebraic and, in fact, there are no non-costant variety morphisms $\mathbb{C} \to \mathbb{C}^\times$ (exercise). The algebraic groups $\mathbb{G}_a$ & $\mathbb{G}_m$ behave very differently.

The situation is better for semisimple Lie algebras/algebraic groups (over $\mathbb{C}$ or, more generally, over $\mathbb{F}$ w. char $\mathbb{F}=0$). And over positive characteristic fields, the relationship is yet more complicated. We'll address this in subsequent notes.