Representation theory of algebraic groups and Lie algebras, 8.5.
1) Distribution algebras in characteristic 0.
2) From homomorphisms of algebraic groups to those of distribution algebras.
3) Rational representations vs modules over distribution algebras.

Ref: [J], Part I, Sec. 7.

This is a follow up to Lec 6.5

1) Let $\mathbb{F}$ be an algebraically closed field, G be a connected algebraic group over $\mathbb{F}$, and $\mathfrak{g}$ be a Lie algebra of G. Recall that the bracket on $\mathfrak{g}$ was described by $[\xi_1,\xi_2]=[\xi_1\otimes\xi_2]\circ C^*$, where $C: G\times G\to G$ is the commutator map $(g_1,g_2)\mapsto g_1g_2g_1^{-1}g_2^{-1}$, see Sec 2 in Lec 6, the proof of Thm there. Equivalently, we can set

$$[\xi_1,\xi_2]=[\xi_1\otimes\xi_2-\xi_2\otimes\xi_1]\circ m^*,$$

where $m: G\times G\to G$ is the multiplication map. This shows that the natural inclusion $\mathfrak{g}=T_1G\hookrightarrow \mathrm{Dist}_1(G)$ is a Lie algebra homomorphism. This gives rise to an algebra homomorphism $\mathcal{U}(\mathfrak{g})\to \mathrm{Dist}_1(G)$.

Thm: if $\mathrm{char}\,\mathbb{F}=0$, then $\mathcal{U}(\mathfrak{g})\xrightarrow{\sim}\mathrm{Dist}_1(G)$.

Sketch of proof (also proving the PBW theorem)

Step 1: Let A be a commutative algebra and $I\subset A$ be an ideal. Set $A_{\geqslant n}:=I^n$ (w. $A_{\geqslant 0}=A$). Then $A=A_{\geqslant 0}\supset A_{\geqslant 1}\supset\ldots$ is a descending algebra filtration (in the sense that $A_{\geqslant i}A_{\geqslant j}\supset A_{\geqslant i+j}$). In this case we can consider the associated graded algebra: $\mathrm{gr}\,A:=\bigoplus_{i=0}^{\infty}A_{\geqslant i}/A_{\geqslant i+1}$

For our filtration coming from the ideal we write $\mathrm{gr}_I A$ insted of $\mathrm{gr}\,A$.

Now suppose X is an affine variety and $\alpha \in X$ is a smooth point. Let $A := \mathbb{F}[X]$, and $I = \mathfrak{m}$ is the maximal ideal of α. Then the natural homomorphism $S(\mathfrak{m}/\mathfrak{m}^2) \longrightarrow \mathrm{gr}_{\mathfrak{m}} A$ is an isomorphism.

Step 2: We apply this to the group G and the point $1 \in G$. The coproduct $\Delta = m^*$ sends $\mathfrak{m}$ to $A \otimes \mathfrak{m} + \mathfrak{m} \otimes A$ and hence $\mathfrak{m}^k$ to $(A \otimes \mathfrak{m} + \mathfrak{m} \otimes A)^k$. It follows that it descends to a coproduct on $\mathrm{gr}_{\mathfrak{m}} A$. Under the identification $\mathrm{gr}_{\mathfrak{m}} A \xrightarrow{\sim} S(\mathfrak{m}/\mathfrak{m}^2)$, we get the coproduct on $S(\mathfrak{m}/\mathfrak{m}^2)$ induced by the algebraic group structure on $(\mathfrak{m}/\mathfrak{m}^2)^* = T_1 G$.

Step 3: Now we investigate the effect of these structures on $\mathrm{Dist}_1(G)$. One can show that the sequence $\mathrm{Dist}_1(G)_{\leq i} := (A/\mathfrak{m}^i)^*$ forms an ascending algebra filtration $\left((\cdot)_{\leq i} (\cdot)_{\leq j} \subset (\cdot)_{\leq i+j} \right)$. The associated graded algebra (defined similarly to the descending case) is then identified with the similar dual of $\mathrm{gr}_{\mathfrak{m}} A = S(\mathfrak{m}/\mathfrak{m}^2)$. So it's the distribution algebra of the algebraic group $T_1 G$. When $\mathrm{char}\, \mathbb{F} = 0$, the latter distribution algebra is $S(T_1 G)$ (with the pairing $S(T_1 G) \otimes S(\mathfrak{m}/\mathfrak{m}^2) \to \mathbb{F}$ induced by the pairing $T_1 G \,(= (\mathfrak{m}/\mathfrak{m}^2)^*) \times \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathbb{F}$.

Step 4: The collection of subspaces $\mathcal{U}(\mathfrak{g})_{\leq d}$ (from Sec 2.2 in Lec 7) forms an ascending filtration on $\mathcal{U}(\mathfrak{g})$. Since $\mathfrak{g}$ lands in $\mathrm{Dist}_1(G)_{\leq 1}$, the homomorphism $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{Dist}_1(G)$ sends $\mathcal{U}(\mathfrak{g})_{\leq d}$ to $\mathrm{Dist}_1(G)_{\leq d}$. This gives a graded algebra homomorphism

$\operatorname{gr} U(\mathfrak{g}) \to \operatorname{gr} \mathrm{Dist}_1(G) = S(\mathfrak{g})$

Note that the easy part of the proof of the PBW theorem, we have $S(\mathfrak{g}) \twoheadrightarrow \operatorname{gr} U(\mathfrak{g})$. The composed homomorphism $S(\mathfrak{g}) \to S(\mathfrak{g})$ is the identity of the degree 1 component, hence the identity. This implies $S(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr} U(\mathfrak{g})$ (the PBW theorem) and also $\operatorname{gr} U(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr} \mathrm{Dist}_1(G)$. The latter implies $U(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Dist}_1(G)$. $\square$

2) From homomorphisms of algebraic groups to those of distribution algebras.

Let G, H be connected algebraic groups, $\varphi: G \to H$ an algebraic group homomorphism. This gives the pullback homomorphism $\varphi^*: \mathbb{F}[H] \to \mathbb{F}[G]$ and hence $\varphi_* := ? \circ \varphi^*: \mathrm{Dist}_1(G) \to \mathrm{Dist}_1(H)$.

Exercise: both φ^* & φ_* are Hopf algebra homomorphisms.

Theorem: Let $\varphi_1, \varphi_2: G \to H$ be two homomorphisms. If $\varphi_{1*} = \varphi_{2*}$, then $\varphi_1 = \varphi_2$.

If $\operatorname{char} \mathbb{F} = 0$, then by Section 1, $\mathrm{Dist}_1(G) = U(\mathfrak{g})$, $\mathrm{Dist}_1(H) = U(\mathfrak{h})$. The homomorphism φ_* is the homomorphism $U(\mathfrak{g}) \to U(\mathfrak{h})$ that is the unique extension of $\varphi := T_1\varphi: \mathfrak{g} \to \mathfrak{h}$ to an algebra homomorphism (exercise). We recover Thm 1 from Section 1.3 in Lec 7.

Proof: Let $A = \mathbb{F}[G]$, $\mathfrak{m} :=$ the maximal ideal of 1 in A. We can consider the completion $\hat{A} := \varprojlim A/\mathfrak{m}^n$. It's an algebra. It is isomorphic to $\mathrm{Dist}_1(G)^*$ (as a vector space, and actually as an algebra).

Now let $B = \mathbb{F}[H]$ and $\hat{B}$ be the similarly defined completion at 1. The homomorphism $\varphi^*: \mathbb{F}[G] \to \mathbb{F}[H]$ induces the homomorphism $\hat{\varphi}^*: \hat{B} \to \hat{A}$. On the other hand, $\varphi_*: \mathrm{Dist}_1(G) \longrightarrow \mathrm{Dist}_1(H)$ gives rise to a linear map $(\varphi_*)^*: \hat{B} \to \hat{A}$. It's left as an exercise to check that $\hat{\varphi}^* = (\varphi_*)^*$.

So, we conclude $\hat{\varphi}_1^* = \hat{\varphi}_2^*$. The following diagram is commutative:

$$\begin{tikzcd} B \arrow[r, "\varphi_i^*"] \arrow[d] & A \arrow[d] \\ \hat{B} \arrow[r, "\hat{\varphi}_i^*"] & \hat{A} \end{tikzcd}$$

Since G is connected ($\Leftrightarrow$ irreducible), $A \hookrightarrow \hat{A}$ ($\Leftrightarrow \bigcap_{i=1}^{\infty} \mathfrak{m}^i = \{0\}$, a special case of the Krull separation thm). So if $\hat{\varphi}_1^* = \hat{\varphi}_2^*$, then $\varphi_1^* = \varphi_2^* \Rightarrow \varphi_1 = \varphi_2$. $\square$

3) Rational representations vs modules over distribution algebras.

It turns out that any rational representation of G is naturally a $\mathrm{Dist}_1(G)$-module. To explain how this works we need the notion of a comodule over a Hopf algebra.

3.1) Comodules.

Recall that a module over an associative (unital) algebra A is a vector space with a linear "action" map $A \otimes_{\mathbb{F}} V \xrightarrow{\alpha} V$ satisfying the following two axioms.

Associativity: the following diagram is commutative.

$$\begin{tikzcd} A\otimes A\otimes V \arrow[r, "\mu\otimes id_V"] \arrow[d, "id_A\otimes\alpha"'] & A\otimes V \arrow[d, "\alpha"] \\ A\otimes V \arrow[r, "\alpha"] & V \end{tikzcd} \qquad (1)$$

Unit: the following diagram is commutative

$$\begin{tikzcd} V=\mathbb{F}\otimes_{\mathbb{F}} V \arrow[rr, "id"] \arrow[dr, "\varepsilon\otimes id_V"'] & & V \\ & A\otimes_{\mathbb{F}} V \arrow[ur, "\alpha"'] & \end{tikzcd} \qquad (2)$$

Definition: A comodule over a coalgebra A (a vector space with a coassociative coproduct and a counit) is a vector space V with a coaction map $\gamma: V \to V\otimes A$ satisfying the coassociativity & counit axioms (the diagrams obtained from (1) & (2) by reversing the arrows).

Suppose for a moment that A is a finite dimensional (associative unital) algebra. Then A^* is a coalgebra. To give an $\mathbb{F}$-linear map $A\otimes_{\mathbb{F}} V \to V$ is the same as to give an $\mathbb{F}$-linear map $V \to V\otimes_{\mathbb{F}} A^*$: via the tensor-Hom adjunction: $\mathrm{Hom}_{\mathbb{F}}(V, V\otimes_{\mathbb{F}} A^*) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(A\otimes_{\mathbb{F}} V, V)$. The former is an action map if and only if the latter is a coaction map. Equivalently, the action map is obtained from the coaction map by $A\otimes V \xrightarrow{id_A\otimes\gamma} A\otimes V\otimes A^* \xrightarrow{\langle\cdot,\cdot\rangle\otimes id_V} V$ (3)

where $\langle\cdot,\cdot\rangle: A\otimes A^* \to \mathbb{F}$ is the pairing $a\otimes f \mapsto \langle f, a\rangle$.

3.2) Rational representations of G vs $\mathbb{F}[G]$-comodules.

Let G be an algebraic group, and V be a rational representation. So, we have a map $V^*\otimes V \to \mathbb{F}[G]$, $\beta\otimes v \mapsto [g\mapsto \langle\beta, gv\rangle]$. This gives rise to an $\mathbb{F}$-linear map $V\to V\otimes \mathbb{F}[G]$, again via the tensor-Hom adjunction.

Exercise: this map is a coaction map, so V is an $\mathbb{F}[G]$-comodule.

Conversely, from an $\mathbb{F}[G]$-comodule structure on V we can get a rational representation: for $v\in V$, $g\in V$, let $\delta(v) = \sum_{i=1}^{k} v_i\otimes f_i$. Then, similarly to (3), set $gv := \sum_{i=1}^{n} f_i(g) v_i$.

Exercise: · this equips V with the structure of a (automatically rational) representation.

· Prove that the two procedures are inverse to each other.

As a conclusion, a rational representation of G is the same thing as an $\mathbb{F}[G]$-comodule.

3.3) From $\mathbb{F}[G]$-comodules to $Dist_1(G)$-modules.

Let V be a rational representation of G, equivalently an $\mathbb{F}[G]$-comodule. We can equip V with a $Dist_1(G)$-module structure similarly to (3): we replace A with $Dist_1(G)$ & A^* with $\mathbb{F}[G]$.

The following claim is a generalization (thx to Sec 1 of this note) of Thm 2 in Sec 1.3 in Lec 7. Assume G is connected.

Theorem: Let V_1, V_2 be rational representations of G & $\varphi: V_1 \to V_2$ be an $\mathbb{F}$-linear map. Then φ is G-linear map $\Leftrightarrow$ φ is $Dist_1(G)$-linear.

Proof: The construction of passing from a rational representation of G to a $Dist_1(G)$-module is "natural", i.e. functorial. So if φ is G-linear, then it's $Dist_1(G)$-linear. Details are left as an exercise.

Note that φ is G-linear iff φ is an $\mathbb{F}[G]$-comodule homomorphism: $\gamma_2 \varphi(v_1) = (\varphi \otimes id_{\mathbb{F}[G]})\gamma_1(v_1)$. Now suppose that φ is $Dist_1(G)$-linear.

Fix bases $v_1^1, \dots, v_1^k$ in V_1, $v_2^1, \dots, v_2^\ell$ in V_2. Let Φ be the matrix of φ in this basis. We can write γ_1, γ_2 as matrices $\Gamma_1 \in Mat_k(\mathbb{F}[G])$, $\Gamma_2 \in Mat_\ell(\mathbb{F}[G])$: if $\Gamma_1 = (\gamma_{ij})$, then we have $\gamma(v_1^i) = \sum_j v_1^j \otimes \gamma_{ij}$.

The condition that φ is a comodule homomorphism translates to the equality $\Phi\Gamma_1 = \Gamma_2\Phi$. The matrices of the action of $\delta \in Dist_1(G)$ on V_1, V_2 are $\delta(\Gamma_1), \delta(\Gamma_2)$ (entrywise evaluation). So the condition that φ is $Dist_1(G)$-linear means $\Phi\delta(\Gamma_1) = \delta(\Gamma_2)\Phi$, $\forall \delta$. Now recall that G is connected. The intersection $\bigcap_{i=1}^{\infty} \mathfrak{m}^i$ for any maximal ideal $\mathfrak{m} \subset \mathbb{F}[G]$ is zero. In particular, if $f \in \mathbb{F}[G]$ satisfies $\delta(f) = 0$, $\forall\, \delta \in Dist_1(G)$, then $f = 0$. So $\Phi\delta(\Gamma_1) = \delta(\Gamma_2)\Phi \Leftrightarrow \delta(\Phi\Gamma_1 - \Gamma_2\Phi) = 0$ and, if this holds for all δ, then $\Phi\Gamma_1 = \Gamma_2\Phi$. This finishes the proof. $\square$

Rem: Not every $\mathrm{Dist}_1(G)$-module comes from a rational representation of G. E.g., consider the additive group $G=\mathbb{G}_a$. As discussed in Example 1 of Sec 2.2 of Lec 6.5, the algebra $\mathrm{Dist}_1(G)$ has basis δ_i, $i\geqslant 0$, and multiplication $\delta_i\delta_j=\binom{i+j}{i}\delta_{i+j}$. One can check that a $\mathrm{Dist}_1(G)$-module comes from a rational representation of G iff δ_i acts by 0 for $i\gg 0$: the element $t\in\mathbb{G}_a$ has to act by $\sum_{i=0}^{\infty}\delta_i t^i$. Not every $\mathrm{Dist}_1(G)$-module has this property: one can check that the following defines a $\mathrm{Dist}_1(G)$-module structure on $\mathbb{F}^2$:

$$\delta_i\mapsto\begin{cases}\begin{pmatrix}1&0\\0&1\end{pmatrix}, & i=0\\ \begin{pmatrix}0&1\\0&0\end{pmatrix}, & i=p^k \text{ for } k>0\\ 0, & \text{else}\end{cases}$$