Representation theory of algebraic groups & Lie algebras, IV

0) Introduction.
1) Representations of $SL_2(\mathbb{F})$ & $\mathfrak{sl}_2(\mathbb{F})$, char $\mathbb{F}=0$.
2) Complement.

0) Let $\mathbb{F}$ be an alg. closed field, G be a connected algebraic group of $\mathbb{F}$ and $\mathfrak{g}$ be the Lie algebra of G. Recall, Sec 1.3 in Lec 5, that G is called simple if it has no proper infinite normal algebraic subgroups and it's noncommutative (the latter condition is similar to the exclusion of $\mathbb{Z}/p\mathbb{Z}$ in the case of finite groups). Similarly, $\mathfrak{g}$ is called simple if it has no proper ideals ($=$subspace $\mathfrak{h}\subset\mathfrak{g}$ w. $[\mathfrak{g},\mathfrak{h}]\subset\mathfrak{h}$) & $\mathfrak{g}$ is not abelian. A connection between these notions is as follows. If $H\subset G$ is an algebraic subgroup, then its Lie algebra $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$. If $H\subset G$ is normal, then $\mathfrak{h}$ is $Ad(G)$-stable, hence $ad(\mathfrak{g})$-stable $\Leftrightarrow$ ideal. So if $\mathfrak{g}$ is simple, then G is simple. The converse is true in characteristic 0 (see the complement section) but may fail in characteristic $p>0$.

Exercise: 1) Check $\mathfrak{sl}_2(\mathbb{F})$ is simple iff char $\mathbb{F}\neq 2$

2*) Check $SL_2(\mathbb{F})$ is always simple.

The algebraic group $SL_2(\mathbb{F})$ and its Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ are the simplest simple algebraic group and Lie algebra (e.g. they have the smallest possible dimension. We will study

1) The representation theory of $\mathfrak{sl}_2(\mathbb{F})$ and $SL_2(\mathbb{F})$ when char $\mathbb{F}=0$ (the latter is essentially a part of the former)

2) The representation theory of $\mathfrak{sl}_2(\mathbb{F})$ for char $\mathbb{F}>2$.

3) The representation theory of $SL_2(\mathbb{F})$ for char $\mathbb{F}>2$ (the case of char $\mathbb{F}=2$ is essentially the same).

These cases already illustrate the essential features of the representation theory of (semi)simple algebraic groups and their Lie algebras (but have none of the complexity of the general case) 1 & 3 are also <u>used</u> to understand the general case.

1) Representations of $SL_2(\mathbb{F})$ & $\mathfrak{sl}_2(\mathbb{F})$, char $\mathbb{F}=0$.

1.1) Universal enveloping algebra.

For now we place no restrictions on $\mathbb{F}$. Let $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{F})$. It has basis $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and the brackets of the basis elements are as follows:

(1) $\qquad [h,e]=2e,\ [h,f]=-2f,\ [e,f]=h.$

It follows that $\mathcal{U}(\mathfrak{g})=\mathbb{F}\langle e,h,f\rangle/([h,e]=2e, [h,f]=-2f, [e,f]=h)$
The PBW theorem (Sec 2.2 in Lec 7) implies that the monomials $f^k h^\ell e^m$ ($k,\ell,m\geqslant 0$) form a basis in $\mathcal{U}(\mathfrak{g})$.

Lemma: In $\mathcal{U}(\mathfrak{g})$ we have the following identities:

$$P(h)e=eP(h+2),\ \forall\ P\in\mathbb{F}[x] \qquad (1)$$

$$P(h)f=fP(h-2),\ \forall\ P\in\mathbb{F}[x] \qquad (2)$$

$$e^m f^n=\sum_{j=0}^{\min(m,n)}\underbrace{\Big(\prod_{i=0}^{j-1}(m-i)(n-i)(i+1)^{-1}\Big)}_{\text{an integer}} f^{n-j}\Big(\prod_{i=0}^{j-1}(h-(m+n)+2j-i)\Big)e^{m-j} \qquad (3)$$

We will need two special cases of (3):

$m=1 \rightsquigarrow \quad ef^n = f^n e + n f^{n-1}(h+1-n)$ (3')

Note that if char $\mathbb{F}=0$, (3) implies

$$\frac{e^n}{n!}\frac{f^n}{n!} = ?e + \binom{h}{n} \left(= \frac{h(h-1)\dots(h-n+1)}{n!}\right) \qquad (3'')$$

Proof of Lemma: (1): $[h,e]=2e \Leftrightarrow he = e(h+2)$. Induction on k shows (1) for $P=x^k$, the general case follows. (2) is similar.

(3'): $ef^n = [e,f]f^{n-1} + fef^{n-1} = hf^{n-1} + fhf^{n-2} + \dots + f^{n-1}h + f^n e = [hf = fh-2f]$
$= f^n e + (h+2(1-n)+h+2(2-n)+\dots+h)f^{n-1} = f^n e + nf^{n-1}(h+1-n)$.

The $m \leq n$ case of (3) follows by induction on m. To handle (3) w. $m \geq n$ one can then use the automorphism $e \mapsto f, f \mapsto e, h \mapsto -h$. The details are left as an exercise. $\square$

1.2) The main result.

Consider the representation of $SL_2(\mathbb{F})$ in homogeneous degree n polynomials, $M(n) := \operatorname{Span}_{\mathbb{F}}(x^n, x^{n-1}y, \dots, y^n)$, $g.f(x,y) := f((x,y)g)$, where we view (x,y) as a row vector. It's rational. The tangent map of this representation is given by

$$e \mapsto x\partial_y,\ f \mapsto y\partial_x,\ h \mapsto x\partial_x - y\partial_y \qquad (4)$$

E.g. $e(x^{n-i}y^i) = \frac{d}{dt}\left[\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}. x^{n-i}y^i\right]_{t=0} = \left[(x,y)\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} = (x, y+tx) \Rightarrow \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}.x^{n-i}y^i = x^{n-i}(y+tx)^i\right] = \frac{d}{dt}\left(x^{n-i}(y+tx)^i\right)\Big|_{t=0} = i x^{n+1-i}y^{i-1} = [x\partial_y](x^{n-i}y^i)$.

Thm: Suppose char $\mathbb{F}=0$. Then for rational representations of $SL_2(\mathbb{F})$/finite dimensional representations of $\mathfrak{sl}_2(\mathbb{F})$, the following claims hold.

1) $n \mapsto M(n)$ gives a bijection between $\mathbb{Z}_{\geq 0}$ and isom. classes of irreps.
2) All representations are completely reducible.

We will treat the case of $\mathfrak{sl}_2$ and deduce the SL_2 case from there.
We will prove (1) in this lecture & (2) in the next.

1.3) Weight decomposition.

At this point $\mathbb{F}$ is still arbitrary. Set $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. Let V be a finite dimensional representation of $\mathfrak{g}$ & $\lambda \in \mathbb{F}$.

Definition: The λ-weight space in V is the generalized eigenspace for h in V with eigenvalue λ: $V_\lambda := \{v \in V \mid \exists\, m > 0 \mid (h-\lambda)^m v = 0\}$
We say: λ is a weight of V if $V_\lambda \neq \{0\}$, a weight vector refers to an element of V_λ for some λ.
Note that, for general reasons, $V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda$.

Example: Consider $V = M(n)$. Then we have $h(x^{n-i}y^i) = [x\partial_x - y\partial_y](x^{n-i}y^i) = (n-2i)x^{n-i}y^i$. It follows that the weights are $n, n-2, \ldots, -n$, we have $M(n)_\lambda = \mathbb{F}x^{n-i}y^i$ for $\lambda = n-2i$ (assuming char $\mathbb{F} = 0$).

Lemma: $eV_\lambda \subset V_{\lambda+2}$, $fV_\lambda \subset V_{\lambda-2}$.
Proof: Let $v \in V_\lambda$. Need to prove $\exists\, m > 0$ s.t. $(h-\lambda-2)^m ev = [(1)$ in Sec 1.1$] = e(h-\lambda)^m v = 0$ for $m \gg 0 \Rightarrow eV \subset V_{\lambda+2}$; $fV \subset V_{\lambda-2}$ follows from (2) there. $\square$

1.4) Highest weight.

Until the end of the lecture, assume char $\mathbb{F}=0$. Define a partial order on $\mathbb{F}$ by $z\leq z'$ if $z'-z\in\mathbb{Z}_{\geq 0}$ (this gives an order precisely because char $\mathbb{F}=0$, which implies $\mathbb{Z}\hookrightarrow\mathbb{F}$).

Definition: a weight of V maximal w.r.t. this order is called a highest weight.

Note that since $\dim V<\infty$, the set of weights of V is finite, so there is a highest weight.

Example: By the previous example, n is the unique highest weight of $M(n)$.

Proposition: Let λ be a highest weight of V & $v\in V_\lambda$. Then

(1) $ev=0$.

(2) $\lambda\in\mathbb{Z}_{\geq 0}$ & $hv=\lambda v$ (i.e. V_λ is an honest eigenspace for h).

Proof: By Lemma in Sec 1.3, $ev\in V_{\lambda+2}=\{0\}$ b/c λ is highest. This proves (1). To prove (2) observe that there's $n>0$ s.t. $\lambda-2n$ is not a weight of V - b/c the set of weights is finite. So $f^n v\in V_{\lambda-2n}=\{0\}$. Consider the vector $\frac{e^n}{n!}\frac{f^n}{n!}v=0$. By (3'') in Sec 1.1, the l.h.s. equals $?ev+\binom{h}{n}v=0$. Since $ev=0$, we get $\binom{h}{n}v=0$ $\Leftrightarrow$ $h(h-1)\dots(h-n+1)v=0$. Since $v\in V_\lambda$ one of the factors kills v, while others must be invertible. Claim (2) follows. $\square$

1.5) Verma modules.

Proposition implies that in every V there is a nonzero element v s.t. $ev=0$, $hv=nv$. We want a "universal" module w. such a vector.

Definition: Let $\lambda\in\mathbb{F}$. By the ==Verma module== $\Delta(\lambda)$ we mean the quotient $\Delta(\lambda)=\mathcal{U}(\mathfrak{g})/I_\lambda$, where $I_\lambda:=\mathcal{U}(\mathfrak{g})(e,h-\lambda)$.

The following proposition describes some properties of $\Delta(\lambda)$.

Proposition: 1) Universal property:

$$\operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\Delta(\lambda),V)\xrightarrow{\sim}\{v\in V\mid ev=0,\ hv=\lambda v \Leftrightarrow I_\lambda v=0\}.$$

2) Basis: if $v_\lambda:=1+I_\lambda\in\Delta(\lambda)$, then the vectors f^iv_λ, $i\geqslant 0$, form a basis in $\Delta(\lambda)$.

3) Submodules: $\Delta(\lambda)$ is simple if $\lambda\notin\mathbb{Z}_{\geqslant 0}$ and has a unique proper submodule, $\operatorname{Span}(f^iv_\lambda\mid i\geqslant\lambda+1)$ else.

Proof: 1) – the isomorphism is given by $\varphi\mapsto\varphi(v_\lambda)$.

2) By the PBW theorem, the elements $f^kh^\ell e^m$ form a basis in $\mathcal{U}(\mathfrak{g})$. It follows that the elements $f^k(h-\lambda)^\ell e^m$ also form a basis. The left ideal I_λ is spanned by these elements w. $\ell>0$ or $m>0$. So the elements $f^k+I_\lambda=f^kv_\lambda$ form a basis in $\mathcal{U}(\mathfrak{g})/I_\lambda$. This shows (2).

3) Note that the elements f^iv_λ form an eigen-basis for h w. pairwise distinct e-values, those are $\lambda-2i$. So any submodule $N\subset\Delta(\lambda)$ is the span of some of these vectors (a Vandermonde determinant argument, exercise). Also if $f^iv_\lambda\in N$, then $f^{i+1}v_\lambda\in N$. So

$N = \mathrm{Span}(f^i v_\lambda \mid i \geq k)$ for some k if $N \neq \{0\}$. If $N \neq \Delta(\lambda)$, then $k > 0$ & $ef^k v_\lambda = 0$. By (3') in Sec. 1.1,

$$ef^k v_\lambda = f^k e v_\lambda + k f^{k-1}(h-(k-1))v_\lambda = 0 + k(\lambda-(k-1))f^{k-1}v_\lambda$$

The left hand side is in N, while the right hand side only lies in N iff it's 0, i.e. $\lambda = k-1$. In particular, for $\lambda \neq k-1$, there is no proper submodule, while for $\lambda = k-1$, $\mathrm{Span}(f^i v_\lambda \mid i \geq \lambda+1)$ is the unique proper submodule. 3) is proved $\square$

1.6) Completion of the classification of irreducibles.

Lie algebra case: For $\lambda \in \mathbb{Z}_{\geq 0}$, set $L(\lambda) = \Delta(\lambda)/\mathrm{Span}(f^i v_\lambda \mid i > \lambda)$. By (3) of Prop'n in Sec 1.5, this is the unique finite dimensional quotient of $\Delta(\lambda)$, and it's irreducible (b/c a proper submodule in $\Delta(\lambda)$ is unique). The modules $L(\lambda)$ are pairwise non-isomorphic b/c $\dim L(\lambda) = \lambda+1$.

Now we show that every irreducible $\mathfrak{sl}_2$-rep'n V is isomorphic to one of $L(\lambda)$. By Prop. in Sec 1.4 $\exists\, v \in V$ s.t. $ev = 0$, $hv = \lambda v$ for $\lambda \in \mathbb{Z}_{\geq 0}$. By (1) of Prop. in Sec 1.5, we get a homomorphism $\Delta(\lambda) \to V$ w. $v_\lambda \mapsto v$. It's nonzero so, since V is irreducible, surjective. But $L(\lambda)$ is the unique finite dim'l quotient of $\Delta(\lambda)$, hence $V \simeq L(\lambda)$.

It remains to establish an isomorphism $M(\lambda) \simeq L(\lambda)$. This is an exercise: can establish it explicitly (via $f^i v_\lambda \mapsto i!\, x^{n-i} y^i$ for $i = 0, \dots n$) or check $M(\lambda)$ is irreducible & use the previous paragraph.

Algebraic group case: Note that every irreducible representation of

$\mathfrak{sl}_2$ is $M(n)$, so comes from a rational representation of SL_2. If V is a representation of SL_2 w. $M(n)\not\subset V$ $\forall n$, then it's reducible over $\mathfrak{sl}_2$. Hence there is an $\mathfrak{sl}_2$-linear embedding $M(n)\hookrightarrow V$ for some n.
By Thm 2 in Sec 1.3 of Lec 6, this embedding is also SL_2-linear, a contradiction.

Complement: algebraic subalgebras.

The goal of this section is to prove the following theorem.

Thm: Suppose $\operatorname{char}\mathbb{F}=0$ & G is a simple algebraic group. Then the Lie algebra $\mathfrak{g}$ is simple.

We note that one uniquely recovers a connected algebraic subgroup $H\subset G$ from T_1H (if $\operatorname{char}\mathbb{F}=0$). For $\mathbb{F}=\mathbb{C}$, this the subgroup of G generated by $\exp(\mathfrak{h})$. If $\mathfrak{h}$ is an ideal, then H is normal (exercise).

The main issue: a Lie subalgebra in $\mathfrak{g}$ may fail to be the tangent space of an algebraic group. However, every (even infinite) intersection of algebraic subgroups of G is algebraic. It follows that for any subalgebra $\mathfrak{h}\subset\mathfrak{g}$ there's the unique minimal Lie algebra of an algebraic group containing $\mathfrak{h}$. It is called the algebraic closure of $\mathfrak{h}$, we denote it by $\overline{\mathfrak{h}}$. See [OV], Sec 3.3 in Ch. 3 - the proofs mirror those of Sec 4.2 in Ch. 1. In particular, by Exer 6 in Sec 4 of Ch. 1 in [OV], if $\mathfrak{h}\subset\mathfrak{g}$ is an ideal, then

$$(*)\qquad [\mathfrak{g},\mathfrak{h}]=[\mathfrak{g},\overline{\mathfrak{h}}].$$

Another fact from [OV] that we need is:

(**) The derived subgroup $(G,G) \subset G$ is a connected algebraic subgroup, Theorem 5 in [OV], Sec 3.4 in Ch. 3. Note that the Lie subalgebra corresponding to (G,G) is $[\mathfrak{g},\mathfrak{g}]$.

Proof of Thm: Suppose $\mathfrak{g}$ is not simple, let $\mathfrak{h}$ be a nonzero ideal. Then $\bar{\mathfrak{h}}$ is an ideal as well by (*). Since G is simple, this ideal coincides with $\mathfrak{g}$. But (G,G) is a normal subgroup and, since we have excluded the commutative groups, we have $(G,G)=G$. So $[\mathfrak{g},\bar{\mathfrak{h}}]=[\mathfrak{g},\mathfrak{g}]$ $=\mathfrak{g}$. So $\mathfrak{h} \supset [\mathfrak{g},\mathfrak{h}]=[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. Contradiction: $\mathfrak{g}$ is simple $\square$