Representation theory of algebraic groups & Lie algebras, V.

1) Casimir element & complete reducibility of $\mathfrak{sl}_2$- & SL_2-reps.
2) Wrap-up on characteristic 0 representation theory of $\mathfrak{sl}_2$.
3) Representations of $\mathfrak{sl}_2$ in characteristic p.
4) Complements.

1.0) Recap.

In Lecture 8 we have classified the finite dimensional irreducible representations of $\mathfrak{g}:=\mathfrak{sl}_2(\mathbb{F})$ w. char $\mathbb{F}=0$: those are $M(n)=\mathrm{Span}_{\mathbb{F}}(x^n, x^{n-1}y, .. y^n)$, $n\in\mathbb{Z}_{\geq 0}$, w. representation given by $e\mapsto x\partial_y$, $h\mapsto x\partial_x - y\partial_y$, $f\mapsto y\partial_x$.

Today we will show that every finite dimensional representation of $\mathfrak{sl}_2$/rational representation of SL_2 is completely reducible finishing our description of these representations.

1.1) Casimir element.

Here is a fundamental observation:

Proposition: The element $C=\frac{1}{2}h^2+h+2fe\in\mathcal{U}(\mathfrak{g})$, Casimir element, is central.

Proof: $\mathcal{U}(\mathfrak{g})$ is generated by e,h,f, so it's enough to check $[e,C]=[h,C]=[f,C]=0$. This is left as an exercise.

In fact, one can understand this element more conceptually - we'll do this in a later lecture. Now we apply C to prove the complete reducibility.

1.2) "Infinitesimal block" decomposition

Let V be a finite dimensional representation of $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{F})$. For $z\in\mathbb{F}$, ~~set~~ let V^z be the generalized eigenspace for C with eigenvalue z: $V^z:=\{v\in V \mid \exists\, m>0 \text{ s.t. } (C-z)^m v=0\}$

We have $V=\bigoplus_{z\in\mathbb{F}} V^z$. The following proposition describes properties of this decomposition.

Proposition: 1) All V^z's are $\mathcal{U}(\mathfrak{g})$-submodules.

2) If $V^z\neq 0$, then $z=\frac{1}{2}n^2+n$ for some $n\geqslant 0$. Moreover, $M(n)$ is the only irreducible constituent of V^z for such z.

Proof: 1): This is because C is central, left as exercise.

2): We claim that C acts on $M(n)$ by the scalar $\frac{1}{2}n^2+n$. It acts by a scalar on $M(n)$ b/c C is central & $M(n)$ is irreducible (Exer. 2.12 in [RT0]). To compute the scalar, say c_n, set $v:=x^n$ so that $hv=nv$, $ev=0$. Hence $c_n v=Cv=(\frac{1}{2}h^2+h+2fe)v=(\frac{1}{2}n^2+n)v \Rightarrow c_n=(\frac{1}{2}n^2+n)$.

Now let $U\subset U'\subset V^z$ be $\mathcal{U}(\mathfrak{g})$-submodules s.t. U'/U is irreducible, i.e. $M(n)$ for some n. Note that z is the only eigenvalue of C on U'/U, i.e. $z=c_n(=\frac{1}{2}n^2+n)$. For each z, there's at most one $n\in\mathbb{Z}_{\geqslant 0}$ with this property. 2) follows. □

1.3) Complete reducibility.

Here we prove (2) of Thm in Sec 1.2 of Lecture 8.

The case of $\mathfrak{sl}_2$: Thanks to the decomposition $V = \bigoplus_z V^z$ and Proposition in Section 1.2, we reduce to proving that $V \simeq M(n)^{\oplus k}$ if $V = V^z$ for $z = \frac{1}{2}n^2 + n \Leftrightarrow V$ admits a filtration

(*) $\{0\} = V^{(0)} \subset V^{(1)} \subset \ldots \subset V^{(k)} = V$ by $\mathfrak{sl}_2$-subreps w. $V^{(i)}/V^{(i-1)} \simeq M(n)$ $\forall i$.

First, we claim $k = \dim V_n$. Indeed, if $V' \subset V$ is a subrepresentation, then from the decomposition $V = \bigoplus_\lambda V_\lambda$ into weight spaces we have

$$V_\lambda / V'_\lambda \xrightarrow{\sim} (V/V')_\lambda, \ \forall \lambda \qquad (1)$$

Setting $\lambda = n$ and using $\dim M(n)_n = 1$, we get $i = \dim V_n^{(i)}$ $\forall i \Rightarrow k = \dim V_n$.

Let $v^1, \ldots v^k$ be a basis in V_n. By Proposition in Sec 1.4 of Lec 8, we have $ev^k = 0$, $hv^k = nv^k$. By (1) of Proposition in Sec 1.5 of Lec 8, $\exists!$ $\mathcal{U}(\mathfrak{g})$-module homomorphism $\Delta(n) \to V$ w. $v_n \mapsto v^i$. By (3) of that Prop'n, $\Delta(n)$ has the unique finite dimensional quotient and, by Sec. 1.6, this quotient is $M(n)$. So $\Delta(n) \to V$ factors through a homomorphism $M(n) \to V$, denote it by φ_i. Consider $\varphi := (\varphi_1, \ldots \varphi_k): M(n)^{\oplus k} \to V$. We claim it's an isomorphism. Thx to (*), $\dim V = \dim M(n)^{\oplus k}$, so it's enough to show that φ is surjective. Let $C = V/\mathrm{im}\,\varphi$. From (1) we deduce $C_n = 0$. But thx to (*), C also admits a filtration w. successive quotients $M(n)$, so $C \neq 0 \Rightarrow C_n \neq 0$. Therefore $C = 0$.

The case of SL_2: Let V be a rational representation. View V as a representation of $\mathfrak{sl}_2$. By what we've proved already, we get an $\mathfrak{sl}_2$-linear isomorphism $\iota: V \xrightarrow{\sim} M(n_1) \oplus \ldots \oplus M(n_k)$ for some $n_1, \ldots n_k \in \mathbb{Z}_{\geqslant 0}$. But the right hand side is a rational rep'n of SL_2. By Thm 2 in Sec 1.3 of Lec 7, ι is SL_2-linear, in particular V is completely reducible.

2) Wrap-up on representations of $\mathfrak{sl}_2$ in characteristic 0.

Some consequences of the classification:

Proposition: for a finite dimensional representation V of $\mathfrak{sl}_2$ we have:

(i) $V = \bigoplus_{i=-n}^{n} V_i$, where h acts on V_i by i.

(ii) $\ker e \subset \bigoplus_{i\geqslant 0} V_i$, $\ker f \subset \bigoplus_{i\leqslant 0} V_i$.

(iii) For each $i \geqslant 0$, the operators $e^i: V_{-i} \to V_i$, $f^i: V_i \to V_{-i}$ are isomorphisms.

Proof: important exercise - use complete reducibility to reduce to $V = M(n)$ and then check by hand.

Rem: There are 3 key techniques in the study of representations of (semi)simple algebraic groups & their Lie algebras. We have seen two of these, they will appear throughout the course.

1) Highest weight theory, roughly, Section 1.3-1.6 of Lec 8 & 1.3 of this lecture.

2) Decomposition into "infinitesimal blocks," roughly, Sec 1.2.

3) Categorical symmetry coming from taking tensor products that we are yet to see.

3) Representations of $\mathfrak{sl}_2$ in char p.

Now take $\mathbb{F}$ of characteristic $p>2$ and set $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. The notion of highest weight no longer makes sense: $z \leqslant z'$ if $z'-z \in \mathbb{Z}_{\geqslant 0}$ is not an order.

However, we have the following crucial observation:

Lemma: The elements $e^p, f^p, h^p - h \in \mathcal{U}(\mathfrak{g})$ are central.

Proof: we need to show that these elements commute with the generators e, h, f of $\mathcal{U}(\mathfrak{g})$. This is done using formulas from Sec 1.1 in Lec 8, e.g. $[e, f^p] = [(3')] = pf^{p-1}(h-p+1) = 0$. The rest is an exercise. $\square$

In the next lecture we will elaborate on these elements more conceptually. For now note that each of $e^p, h^p - h, f^p$ act on every irreducible finite dimensional module by scalars to be denoted by χ_1, χ_2, χ_3 and let $\chi := (\chi_1, \chi_2, \chi_3)$. In fact, for each triple there is an irreducible module giving this triple (see the complement section) but we can reduce to the 2 special values and one family:

- $(0,0,0)$
- $(0,0,1)$
- $(0,a,0)$, $a \neq 0$.

We will elaborate on the reduction in the next lecture. For now we will analyze these 3 cases.

Case $\chi = 0$: the irreducible representations are exactly $M(i)$, $i = 0, \dots p-1$.

Proof: Let V be a $\mathcal{U}(\mathfrak{g})$-module annihilated by the central elements $e^p, f^p, h^p - h$.

Step 1: $h^p - h = \prod_{i \in \mathbb{F}_p} (h-i)$ acts by 0 on V. So $V = \bigoplus_{\lambda \in \mathbb{F}_p} V_\lambda$, V_λ is the λ-eigenspace for h. The element e acts by a nilpotent operator.

Since $eV_\lambda \subset V_{\lambda+2}$ (Lemma in Sec 1.3 in Lec 9), $\ker e = \bigoplus_\lambda (V_\lambda \cap \ker e)$
$\Rightarrow \exists\, v \in V$ w. $ev=0$, $hv=\lambda v$ for some $\lambda \in \mathbb{F}_p$. Besides, $f^p v=0$.

Step 2: The Verma module $\Delta(\lambda) = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})(e, h-\lambda)$ still makes sense, and 1) & 2) of Prop'n in Sec 1.5 of Lec 8 hold. In particular, $\exists!$ homom'm $\varphi: \Delta(\lambda) \to V$ w. $v_\lambda \mapsto v$. Note that, since f^p is central, $f^p\Delta(\lambda) \subset \Delta(\lambda)$ is a submodule. From $f^p v=0$, we see that φ factors through $\underline{\Delta}^0(\lambda) := \Delta(\lambda)/f^p\Delta(\lambda)$, known as the baby Verma module.

Step 3: Let $\underline{v}_\lambda$ be the image of v_λ in $\underline{\Delta}^0(\lambda)$. Then the elements $\underline{v}_\lambda, f\underline{v}_\lambda, \dots, f^{p-1}\underline{v}_\lambda$ form a basis in $\underline{\Delta}^0(\lambda)$ & $hf^i\underline{v}_\lambda = (\lambda-2i)\underline{v}_\lambda$. From here we can analyze the submodules of $\underline{\Delta}^0(\lambda)$ similarly to what was done for the usual Verma modules, (3) of Prop'n in Sec 1.5 of Lec 8.

Exercise: $\underline{\Delta}^0(\lambda)$ is irreducible if $\lambda = p-1$ ($\in \mathbb{F}_p$) and has the unique proper submodule else. This submodule is $\operatorname{Span}_{\mathbb{F}}(f^i\underline{v}_\lambda \mid \lambda < i \leq p-1)$.

Step 4: We then proceed as in Section 1.6 of Lec 8. The details are left as an exercise. $\square$

Rem: $\underline{\Delta}^0(\lambda)$ w. $\lambda \neq p-1$ is <u>not</u> completely reducible.

Case $\chi = (0,0,1)$: in this case we have $\frac{p+1}{2}$ irreps, all have $\dim = p$.

Proof: Define $\underline{\Delta}^1(\lambda) := \Delta(\lambda)/(f^p-1)\Delta(\lambda)$. We can analyze these $\mathcal{U}(\mathfrak{g})$-modules similarly to Steps 3,4 of the previous case.

Exercise: $\underline{\Delta}^1(\lambda)$ is irreducible $\forall \lambda \in \mathbb{F}_p$. Moreover, every irreducible $\mathcal{U}(\mathfrak{g})$-module annihilated by e^p, h^p-h, f^p-1 is isomorphic to one of $\underline{\Delta}^1(\lambda)$.

But unlike what we've seen before some of $\underline{\Delta}^\chi(\lambda)$'s are isomorphic - we claim $\underline{\Delta}^1(\lambda) \simeq \underline{\Delta}^1(\lambda') \Leftrightarrow \lambda + \lambda' = -2$. The following claim left as an exercise is proved along the lines of the proof of (3) of Proposition in Sec 1.5 of Lec 8:

$$hf^{\lambda+1}\underline{v}_\lambda = -(\lambda+2)\underline{v}_\lambda, \quad ef^{\lambda+1}\underline{v}_\lambda = 0 \quad (\lambda \in \{0,1,..,p-2\})$$

This gives a nonzero homomorphism $\Delta(-\lambda-2) \longrightarrow \underline{\Delta}^1(\lambda)$, which factors $\underline{\Delta}^1(-\lambda-2) \longrightarrow \underline{\Delta}^1(\lambda)$, which is an isomorphism b/c both modules are irreducible.

And if $\lambda + \lambda' \neq -2$, then $\underline{\Delta}^1(\lambda) \not\simeq \underline{\Delta}^1(\lambda')$. Indeed, C acts on $\underline{\Delta}^1(\lambda)$ by $\frac{1}{2}\lambda^2+\lambda$ (apply C to $\underline{v}_\lambda$) and $\frac{1}{2}\lambda^2+\lambda = \frac{1}{2}\lambda'^2+\lambda' \Leftrightarrow \lambda=\lambda'$ or $\lambda+\lambda'=-2$.

This completes the proof. $\square$

Exercise: Let $a \in \mathbb{F}\setminus\{0\}$ and $\lambda_1, .., \lambda_p$ be the roots of $x^p - x - a = 0$. Then $\underline{\Delta}^0(\lambda_i) = \Delta(\lambda_i)/f^p\Delta(\lambda_i)$, $i=1,..,p$, are exactly the pairwise non-isomorphic irreducible $\mathcal{U}(\mathfrak{g})$ modules annihilated by e^p, f^p, h^p-h-a.

4) Complement: p-center & central reduction.

Definition: • By the p-center of $\mathcal{U}(\mathfrak{g})$ we mean the subalgebra generated by e^p, h^p-h, f^p. Denote it by Z_p.

• Let $\chi=(\chi_1,\chi_2,\chi_3)\in\mathbb{F}^3$. By the p-central reduction we mean the algebra

$$\mathcal{U}^\chi = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})(e^p-\chi_1,\ h^p-h-\chi_2,\ f^p-\chi_3)$$

(it's an algebra because the generators of the left ideal we mod out are central and so the left ideal is, in fact, 2-sided).

The following proposition describes basic properties of Z_p.

Proposition: 1) The generators e^p, h^p-h, f^p of Z_p are free, i.e. Z_p is isomorphic to the algebra of polynomials in 3 variables.

2) $\mathcal{U}(\mathfrak{g})$ is a free Z_p-module w. basis $f^k h^\ell e^m$ w. $0\leq k,\ell,m\leq p-1$, so of rank p^3.

Proof: The elements

$$f^{k_1}(f^p)^{k_2}h^{\ell_1}(h^p-h)^{\ell_2}e^{m_1}(e^p)^{m_2} \qquad (*)$$

w. $0\leq k_1,\ell_1,m_1\leq p-1$ & $k_2,\ell_2,m_2\geq 0$ form a basis in $\mathcal{U}(\mathfrak{g})$ – they are obtained from the PBW basis by applying a unitriangular transformation. Note that e^p, h^p-h, f^p are central, so $(*) = f^{k_1}h^{\ell_1}e^{m_1}\underbrace{[(f^p)^{k_2}(h^p-h)^{\ell_2}(e^p)^{m_2}]}_{\in Z_p}$. Both claims follow. □

Corollary: $\dim \mathcal{U}^\chi = p^3$ $\forall \chi\in\mathbb{F}^3$, exercise.

Recall that every central element acts on a finite dimensional irrep by a scalar. This gives rise to a bijection

$$\underbrace{\mathrm{Irr}_{fd}}_{\text{iso classes of fin. dim'l irreps}}(\mathcal{U}(\mathfrak{g})) = \coprod_{\chi} \mathrm{Irr}(\mathcal{U}^\chi)$$

where $\mathrm{Irr}(U^\chi)$ embeds into $\mathrm{Irr}_{fd}(U(\mathfrak{g}))$ via composing with the epimorphism $U(\mathfrak{g}) \twoheadrightarrow U^\chi$ (and so the image consists of all irreps, where $e^p, h^p - h, f^p$ act by χ_1, χ_2, χ_3, respectively.

By Corollary, $U^\chi \neq \{0\}$ so $\mathrm{Irr}(U^\chi) \neq \emptyset$. So unlike in char 0 case, there are uncountably many $\mathfrak{g}$-irreps.