1.0) Recap.

In Lecture 8 we have classified the finite dimensional irreducible representations of $g:=\mathfrak{sl}_2(F)$ w. char $F=0$: those are $M(n)=\text{Span}_F \{x^n, x^{n-1}y, \ldots, y^n\}$, $n\in \mathbb{Z}_+$, w. representation given by $e\mapsto x\partial_y$, $h\mapsto x\partial_x - y\partial_y$, $f\mapsto y\partial_x$.

Today we will show that every finite dimensional representation of $\mathfrak{sl}_2$/rational representation of $SL_2$ is completely reducible, finishing our description of these representations.

1.1) Casimir element.

Here is a fundamental observation:

**Proposition:** The element $C = \frac{1}{2} h^2 + h + f \in U(g)$, Casimir element, is central.

Proof: $U(g)$ is generated by $e, h, f$, so it's enough to check $[e, C] = [h, C] = [f, C] = 0$. This is left as an exercise.

In fact, one can understand this element more conceptually - we'll do this in a later lecture. Now we apply $C$ to prove
the complete reducibility.

1.2) "Infinitesimal block" decomposition

Let $V$ be a finite dimensional representation of $g = \mathfrak{gl}_2(F)$. For $z \in F$, set $V^z$ be the generalized eigenspace for $C$ with eigenvalue $z$:

$$V^z := \{ v \in V | \exists m \geq 0 \text{ s.t. } (C-z)^m v = 0 \}$$

We have $V = \bigoplus_{z \in F} V^z$. The following proposition describes properties of this decomposition.

**Proposition:** 1) All $V^z$'s are $U(g)$-submodules.

2) If $V^z \neq 0$, then $z = \frac{1}{2} n^2 + n$ for some $n \geq 0$. Moreover, $M(n)$ is the only irreducible constituent of $V^z$ for such $z$.

**Proof:** 1): This is because $C$ is central, left as exercise.

2): We claim that $C$ acts on $M(n)$ by the scalar $\frac{1}{2} n^3 + n$. It acts by a scalar on $M(n)$ b/c $C$ is central & $M(n)$ is irreducible (Exer. 2.12 in [RT01]). To compute the scalar, say $c_n$, let $v = x^n$ so that $hv = mv$, $ev = 0$. Hence $c_n v = C v = (\frac{1}{2} h^2 + h + 2fe) v = (\frac{1}{2} n^2 + n) v \Rightarrow c_n = (\frac{1}{2} n^3 + n)$.

Now let $U \subset U' \subset V^z$ be $U(g)$-submodules s.t. $U/U'$ is irreducible, i.e. $M(n)$ for some $n$. Note that $z$ is the only eigenvalue of $C$ on $U/U'$, i.e. $z = c_n = (\frac{1}{2} n^2 + n)$. For each $z$, there's at most one $n \in \mathbb{Z}_{>0}$ with this property. 2) follows. \(\square\)

1.3) Complete reducibility

Here we prove (2) of Thm in Sec 1.2 of Lecture 8.
The case of $S_n^2$: Thanks to the decomposition $V = \bigoplus_a V^a$ and Proposition in Section 1.2, we reduce to proving that $V \cong M(n)^{\oplus k}$ if $V = V^2$ for $z = \frac{1}{2}n^2 + n \iff V$ admits a filtration

\[ \{0\} = V^0 \subset V^1 \subset \cdots \subset V^{(k)} = V \] by $S_2$-subreps w. $V^{(i)}/V^{(i-1)} \cong M(n)$ $\forall i$.

First, we claim $k = \dim V_n$. Indeed, if $V \subset V$ is a subrepresentation, then from the decomposition $V = \bigoplus V_\lambda$ into weight spaces we have

\[ V_\lambda/V_\lambda' \cong (V/V')_{\lambda}, \quad \forall \lambda \quad (1) \]

Setting $\lambda = n$ and using $\dim M(n)_n = 1$, we get $i = \dim V_n^{(i)}$ $\forall i \Rightarrow k = \dim V_n$.

Let $v^0, \ldots, v^k$ be a basis in $V_n$. By Proposition in Sec 1.6 of Lec 8, we have $ev^0 = 0, hv^k = n v^k$. By (1) of Proposition in Sec 1.5 of Lec 8, $\exists! U(q)$-module homomorphism $\Delta(n) \rightarrow V$ w. $v_i \mapsto v^i$. By (3) of that Prop'n, $\Delta(n)$ has the unique finite dimensional quotient and, by Sec. 1.6, this quotient is $M(n)$. So $\Delta(n) \rightarrow V$ factors through a homomorphism $M(n) \rightarrow V$, denote it by $\phi$. Consider $\phi: (\varphi_1, \ldots, \varphi_k): M(n)^{\oplus k} \rightarrow V$.

We claim it's an isomorphism. Thx to (*), $\dim V = \dim M(n)^{\oplus k}$, so it's enough to show that $\phi$ is surjective. Let $C = V/\text{im} \phi$. From (1) we deduce $C \cong 0$. But thx to (*), $C$ also admits a filtration w. successive quotients $M(n)$, so $C \neq 0 \Rightarrow C_n \neq 0$. Therefore $C = 0$.

The case of $SL_2$: Let $V$ be a rational representation. View $V$ as a representation of $S_n^2$. By what we’ve proved already, we get an $S_n^2$-linear isomorphism $\iota: V \rightarrow M(n)^{\oplus} \oplus M(n_\lambda)$ for some $n, n_\lambda \in \mathbb{Z}_{>0}$. But the right hand side is a rational rep'n of $S_n^2$. By Thm 2 in Sec 1.3 of Lec 7, it is $S_n^2$-linear, in particular $V$ is completely reducible.
2) Wrap-up on representations of $\mathfrak{g}_2$ in characteristic 0.

Some consequences of the classification:

**Proposition:** for a finite dimensional representation $V$ of $\mathfrak{g}_2$ we have:

(i) $V = \bigoplus_{i=n}^{\infty} V_i$, where $h$ acts on $V_i$ by $i$.

(ii) $\ker e < \bigoplus_{i=0}^\infty V_i$, $\ker f < \bigoplus_{i=0}^\infty V_i$.

(iii) For each $i \geq 0$, the operators $e_i : V_i \to V_i$, $f_i : V_i \to V_i$ are isomorphisms.

Proof: important exercise — use complete reducibility to reduce to $V = M(n)$ and then check by hand.

**Rem:** There are 3 key techniques in the study of representations of (semi) simple algebraic groups & their Lie algebras. We have seen two of these, they will appear throughout the course.

1) **Highest weight theory**, roughly, Section 1.3-1.6 of Lec 8 & 1.3 of this lecture.

2) **Decomposition into "infinitesimal blocks,"** roughly, Sec 1.2.

3) **Categorical symmetry coming from taking tensor products that we are yet to see.**

3) Representations of $\mathfrak{g}_2$ in char $p$

Now take $\mathbb{F}$ of characteristic $p > 2$ and set $g = \mathfrak{g}_2(\mathbb{F})$. The notion of highest weight no longer makes sense: $\mathbb{Z} \leq \mathbb{Z}^+$ if $2 \cdot \mathbb{Z} \in \mathbb{Z}_+$ is not an order. However, we have the following crucial observation:
Lemma: The elements $e,f,h-h \in U(g)$ are central.

Proof: We need to show that these elements commute with the generators $e,f,h$ of $U(g)$. This is done using formulas from Sec. 1.1 in Lec. 8, e.g. $[e,f]=[(\lambda')]=pf^{-1}(h-p+1)=0$. The rest is an exercise. □

In the next lecture we will elaborate on these elements more conceptually. For now note that each of $e, h-h, f$ act on every irreducible finite dimensional module by scalars to be denoted by $X_1, X_2, X_3$ and let $X = (X_1, X_2, X_3)$. In fact, for each triple there is an irreducible module giving this triple (see the complement section) but we can reduce to the 2 special values and one family:

- $(0,0,0)$
- $(0,0,1)$
- $(0,0,0), a \neq 0$.

We will elaborate on the reduction in the next lecture. For now we will analyze these 3 cases.

Case $X=0$: the irreducible representations are exactly $M(i), i=0,\ldots,p-1$.

Proof: Let $V$ be a $U(g)$-module annihilated by the central elements $e, f, h-h$.

Step 1: $h-h = \sum (h-i)$ acts by 0 on $V$. So $V = \bigoplus_{\lambda \in \mathbb{F}_p^*} V_{\lambda}$, $V_{\lambda}$ is the $\lambda$-eigenspace for $h$. The element $e$ acts by a nilpotent operator.
Since \( e V_\lambda < V_\lambda \) (Lemma in Sec 1.3 in Lec 8), \( \ker e = \bigoplus_{\lambda \in \mathbb{F}_p} (V_\lambda \cap \ker e) \)
\( \Rightarrow \exists v \in V \) w. \( ev = 0 \), \( hv = \lambda v \) for some \( \lambda \in \mathbb{F}_p \). Besides, \( f^p v = 0 \).

**Step 2:** The Verma module \( \Delta(\lambda) = U(g) / U(g) (e, h - \lambda) \) still makes sense, and 1) & 2) of Prop’n in Sec 1.5 of Lec 8 hold. In particular, \( \exists \)! homom. \( \phi: \Delta(\lambda) \to V \) w. \( \phi_\lambda \to v_\lambda \). Note that, since \( f^p \) is central, \( f^p \Delta(\lambda) \subset \Delta(\lambda) \) is a submodule. From \( f^p v = 0 \), we see that \( \phi \) factors through \( \Delta^0(\lambda) := \Delta(\lambda) / f^p \Delta(\lambda) \), known as the *baby Verma module*.

**Step 3:** Let \( \bar{v}_\lambda \) be the image of \( v_\lambda \) in \( \Delta^0(\lambda) \). Then the elements \( \bar{v}_\lambda, f \bar{v}_\lambda, \ldots, f^{p-1} \bar{v}_\lambda \) form a basis in \( \Delta^0(\lambda) \) & \( h f^i \bar{v}_\lambda = (\lambda - 2i) \bar{v}_\lambda \). From here we can analyze the submodules of \( \Delta^0(\lambda) \) similarly to what was done for the usual Verma modules, (3) of Prop’n in Sec 1.5 of Lec 8.

**Exercise:** \( \Delta^0(\lambda) \) is irreducible if \( \lambda = p - 1 (\in \mathbb{F}_p) \) and has the unique proper submodule else. This submodule is \( \text{Span}_e \{ f^i \bar{v}_\lambda \mid \lambda < i \leq p - 1 \} \).

**Step 4:** We then proceed as in Section 1.6 of Lec 8. The details are left as an exercise.

**Rem:** \( \Delta^0(\lambda) \) w. \( \lambda \neq p - 1 \) is not completely reducible.

**Case \( \delta = (0, 0, 1) \):** in this case we have \( \frac{p + 1}{2} \) irreps, all have \( \text{dim} = p \).
Proof: Define $\Delta^I (\lambda) = \Delta(\lambda) \cap (f^p, h) \Delta(\lambda)$. We can analyze these $U(g)$-modules similarly to Steps 3, 4 of the previous case.

Exercise: $\Delta^I (\lambda)$ is irreducible $\forall \lambda \in F_p$. Moreover, every irreducible $U(g)$-module annihilated by $e^p, h^p - h, f^p - 1$ is isomorphic to one of $\Delta^I (\lambda)$.

But unlike what we've seen before some of $\Delta^I (\lambda)$'s are isomorphic - we claim $\Delta^I (\lambda) \cong \Delta^I (\lambda') \iff \lambda + \lambda' = -2$. The following claim left as an exercise is proved along the lines of the proof of (3) of Proposition in Sec 1.5 of Lec 8:

$$hf^{\lambda+1} \xi_{\lambda} = -(\lambda+2) \xi_{\lambda}, \quad ef^{\lambda+1} \xi_{\lambda} = 0 \quad (\lambda \in \mathbb{F}_p)$$

This gives a nonzero homomorphism $\Delta(-\lambda-2) \to \Delta^I (\lambda)$, which factors $\Delta(-\lambda-2) \to \Delta^I (\lambda)$, which is an isomorphism b/c both modules are irreducible.

And if $\lambda + \lambda' = -2$, then $\Delta^I (\lambda) \not\cong \Delta^I (\lambda')$. Indeed, $C$ acts on $\Delta^I (\lambda)$ by $\frac{1}{2} \lambda + 1$ (apply $C$ to $\xi_{\lambda}$) and $\frac{1}{2} \lambda + 1 = \frac{1}{2} \lambda' + \lambda' \iff \lambda = \lambda'$ or $\lambda + \lambda' = -2$.

This completes the proof. \qed

Exercise: Let $a \in \mathbb{F} \setminus \{0, 1\}$ and $\lambda_1, \ldots, \lambda_p$ be the roots of $x^p - x - a = 0$. Then $\Delta^0 (\lambda_i) = \Delta (\lambda_i) / f^p \Delta (\lambda_i), \ i = 1, \ldots, p$, are exactly the pairwise non-isomorphic irreducible $U(g)$ modules annihilated by $e^p, f^p, h^p - h - a$.

4) Complement: $p$-center & central reduction

Definition: By the $p$-center of $U(g)$ we mean the subalgebra generated by $e^p, h^p - h, f^p$. Denote it by $Z_p$. 

\textit{Theorem:}
Let $X = (x_1, x_2, x_3) \in \mathbb{F}^3$. By the \textit{p-central reduction}, we mean the algebra

$$U^X = U(g)/\langle U(g)(e^p - x_1, h^p - x_2, f^p - x_3) \rangle$$

(it's an algebra because the generators of the left ideal we mod out are central and so the left ideal is, in fact, 2-sided).

The following proposition describes basic properties of $\mathbb{Z}_p$.

**Proposition:**
1. The generators $e^p, h^p, f^p$ of $\mathbb{Z}_p$ are free, i.e. $\mathbb{Z}_p$ is isomorphic to the algebra of polynomials in 3 variables.
2. $U(g)$ is a free $\mathbb{Z}_p$-module w. basis $f^k h^l e^m$ w. $0 \leq k, l, m < p$.

**Proof:** The elements

$$f^{k_1} (f^p)^{k_2} h^{l_1} (h^p)^{l_2} e^{m_1} (e^p)^{m_2}$$

w. $0 \leq k, l, m < p - 1$ & $k_1, l_1, m_1 \geq 0$ form a basis in $U(g)$ — they are obtained from the PBW basis by applying a unitriangular transformation. Note that $e^p, h^p, f^p$ are central, so $f^{k_1} h^{l_1} e^{m_1} (f^p)^{k_2} (h^p)^{l_2} e^{m_2} (e^p)^{m_2} \in \mathbb{Z}_p$

Both claims follow.

**Corollary:** $\dim U^X = p^3 \forall X \in \mathbb{F}^3$.

Recall that every central element acts on a finite dimensional irrep by a scalar. This gives rise to a bijection

$$\text{Irr}_{\mathbb{Z}_p}(U(g)) \cong \bigcup_{X \in \mathbb{F}^3} \text{Irr}(U^X)$$

iso classes of fin. dim. irreps
where $\text{Irr}(U^X)$ embeds into $\text{Irr}_C(U(\alpha))$ via composing with the epimorphism $U(\alpha) \rightarrow U^X$ (and so the image consists of all irreps, where $e^p, h^p, f^p$ act by $X_1, X_2, X_3$, respectively.

By Corollary, $U^X \neq \{0\}$ so $\text{Irr}(U^X) \neq \emptyset$. So unlike in char 0 case, there are uncountably many $\alpha$-irreps.