REPRESENTATIONS OF SL_2 AND $\mathfrak{sl}_2$

1. Introduction

Let $\mathbb{F}$ be an algebraically closed field, and let G be a connected algebraic group over $\mathbb{F}$ with Lie algebra $\mathfrak{g}$. Recall, [2, Section 1.4], that G is called *simple* if it has no proper infinite normal algebraic subgroups and is noncommutative. The latter condition is similar in spirit to excluding $\mathbb{Z}/p\mathbb{Z}$ in the case of finite groups.

Now we explain what "simple" means for Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. We say that a subspace $\mathfrak{h} \subset \mathfrak{g}$ is an *ideal*, if $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{g}, y \in \mathfrak{h}$. These are exactly the kernels of Lie algebra homomorphisms from $\mathfrak{g}$. We say that $\mathfrak{g}$ is *simple* if it has no proper (=different from $\{0\}$ and $\mathfrak{g}$) ideals and is not abelian.

If $H \subset G$ is an algebraic subgroup, then its Lie algebra $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, the Lie algebra of G, see (2) of [2, Theorem 2.18]. If H is normal in G, then $\mathfrak{h}$ is an $\mathrm{Ad}(G)$-stable subspace. Since the adjoint $\mathfrak{g}$-action is obtained by differentiating the adjoint G-action, [2, Lemma 2.28], $\mathfrak{h}$ is an ideal. Thus, if $\mathfrak{g}$ is simple, then G is simple. The converse holds in characteristic zero but may fail in positive characteristic. For the implication in characteristic 0 the reader is referred to [OV, §4.1.3]. The idea is as follows: once we have a proper ideal $\mathfrak{h} \subset \mathfrak{g}$ which may fail to be the Lie algebra of an algebraic subgroup of G there is still a proper ideal, say $\mathfrak{h}'$, that corresponds to an algebraic subgroup, H'. Every ideal is stable under the adjoint action of G, thanks to the exponential map, see [2, Sec. 2.7]. Hence H' has to be normal.

In positive characteristic it may happen that the Lie algebra is not simple, while the algebraic group is, as the following exercise illustrates.

Exercise 1.1. Check that $\mathfrak{sl}_2(\mathbb{F})$ is simple if $\operatorname{char} \mathbb{F} \neq 2$. Further, check that $\mathrm{SL}_2(\mathbb{F})$ is always simple.

The algebraic group $\mathrm{SL}_2(\mathbb{F})$ and its Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ are the easiest examples of a simple algebraic group and Lie algebra, for instance, they have the smallest possible dimension. In this lecture we will study the following topics.

(1) The representation theory of $\mathrm{SL}_2(\mathbb{F})$ and $\mathfrak{sl}_2(\mathbb{F})$ when $\operatorname{char} \mathbb{F} = 0$; the latter is essentially part of the former.
(2) The representation theory of $\mathfrak{sl}_2(\mathbb{F})$ for $\operatorname{char} \mathbb{F} > 2$.
(3) The representation theory of $\mathrm{SL}_2(\mathbb{F})$ for $\operatorname{char} \mathbb{F} > 2$ (the case $\operatorname{char} \mathbb{F} = 2$ is essentially the same).

These cases already illustrate the essential features of the representation theory of (semi)simple algebraic groups and their Lie algebras, but have none of the complexity of the general case. Representations of $\mathrm{SL}_2(\mathbb{F})$ and $\mathfrak{sl}_2(\mathbb{F})$ are also used to understand the general case.

2. Representations of $\mathfrak{sl}_2(\mathbb{F})$ for $\operatorname{char} \mathbb{F} = 0$

2.1. **Universal enveloping algebra.** For now, we place no restrictions on the base field $\mathbb{F}$. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. This Lie algebra has a basis e, f, h, and the brackets of the basis elements are as follows:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

It follows that the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is generated by e, f, h with the relations

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h.$$

The PBW (Poincare-Birkhoff-Witt) theorem, [2, Thm. 3.4], implies that the monomials $f^a h^b e^c$ for $a, b, c \geqslant 0$ form a basis in $U(\mathfrak{g})$.

Lemma 2.1. *In $U(\mathfrak{g})$, we have the following identities (where P is any polynomial in one variable and k, m are any nonnegative integers):*

$$P(h)e^m = e^m P(h+2m), \tag{1}$$

$$P(h)f^k = f^k P(h-2k), \tag{2}$$

$$e^m f^k = \sum_{j=0}^{\min(m,k)} \left(\prod_{i=0}^{j-1} \frac{(m-i)(k-i)}{i+1} \right) f^{k-j} \left(\prod_{i=0}^{j-1} (h - m - k + 2j - i) \right) e^{m-j}. \tag{3}$$

Note that in the case of $m = 1$ (3) reads:

$$ef^k = f^k e + k f^{k-1}(h + 1 - k). \tag{4}$$

Proof of Lemma 2.1. For (1), use induction on k to show that $he^k = e^k(h + 2k)$; the general case follows. The proof of (2) is similar. For (3), use induction on k to handle (4), then the induction on m to handle the general case. The details are left as an exercise. ☐

Note that if $\operatorname{char} \mathbb{F} = 0$, then (3) implies

$$\frac{e^n}{n!} \frac{f^n}{n!} - \frac{1}{n!} \prod_{j=0}^{n-1} (h - j) \in U(\mathfrak{g})e. \tag{5}$$

2.2. **The main result.** Consider the representation of $\mathrm{SL}_2(\mathbb{F})$ in the space of homogeneous degree n two variable polynomials

$$M(n) = \operatorname{Span}\{x^i y^{n-i} \mid 0 \le i \le n\}, g.f(x, y) := f((x, y)g),$$

where in the definition of the action we view (x, y) as a row vector so that we can multiply it by a matrix from the right. It is clear that $M(n)$ is a rational representation.

Exercise 2.2. The representation of $\mathfrak{sl}_2(\mathbb{F})$ obtained by differentiating the representation of $\mathrm{SL}_2(\mathbb{F})$ in $M(n)$, see [2, Sec. 2.6], is given by

$$e \mapsto x\frac{\partial}{\partial y}, \quad f \mapsto y\frac{\partial}{\partial x}, \quad h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Example 2.3. For $n = 2$, the representation of $\mathfrak{sl}_2(\mathbb{F})$ is as follows:

$$\begin{array}{lll} e.x^2 = 0, & e.xy = x^2, & e.y^2 = 2xy, \\ f.x^2 = 2xy, & f.xy = y^2, & f.y^2 = 0, \\ h.x^2 = 2x^2, & h.xy = 0, & h.y^2 = -2y^2. \end{array}$$

Here is the main theorem of this part.

Theorem 2.4. *Suppose that $\mathbb{F}$ is an algebraically closed field of characteristic 0. Then for the rational representations of $\mathrm{SL}_2(\mathbb{F})$ and for the finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{F})$, the following claims hold:*

(1) *The map* $n \mapsto M(n)$ *gives a bijection between* $\mathbb{Z}_{\geqslant 0}$ *and the set of isomorphism classes of irreducible representations.*
(2) *All representations in the classes mentioned above are completely reducible.*

We will treat the case of $\mathfrak{sl}_2$ and deduce the SL_2 case from there.

Here is an important consequence of the classification that is going to be used in the subsequent lectures.

Corollary 2.5. *Let* $\mathbb{F}$ *be an algebraically closed field of characteristic* 0 *and let* V *be a finite dimensional representation of* $\mathfrak{sl}_2(\mathbb{F})$. *Then the following claims hold:*

(1) V *splits as the direct sum* $\bigoplus_{i\in\mathbb{Z}} V_i$, *where* h *acts on* V_i *by the multiplication by* i.
(2) $\ker e \subset \bigoplus_{i\geqslant 0} V_i$ *and* $\ker f \subset \bigoplus_{i\leqslant 0} V_i$.
(3) *For each* $i \geqslant 0$, *the element* e^i *gives an isomorphism* $V_{-i} \to V_i$, *while* f^i *gives an isomorphism* $V_i \xrightarrow{\sim} V_{-i}$.

Proof. Theorem 2.4 reduces the proof to the case when $V = M(n)$. Here (1)-(3) follow from a direct check using Exercise 2.2, details of this check are also left as an exercise. □

2.3. **Weight decomposition.** At this point, $\mathbb{F}$ is still an arbitrary field. Set $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. Let V be a finite-dimensional representation of $\mathfrak{g}$.

Definition 2.6. Let $\lambda \in \mathbb{F}$. The λ-*weight space* in V is the generalized eigenspace for h in V with eigenvalue λ:

$$V_\lambda = \{v \in V \mid (h - \lambda\,\mathrm{id})^m v = 0 \text{ for some } m \geqslant 1\}.$$

We say λ is a *weight* of V if $V_\lambda \neq 0$, and a *weight vector* refers to a nonzero element of V_λ for some λ.

Example 2.7. Consider $V = M(n)$. Then we have

$$h.x^i y^{n-i} = (n-2i)x^i y^{n-i},$$

so the weights are $n, n-2, \ldots, -n \in \mathbb{F}$. Since

$$M(n) = \bigoplus_{i=0}^{n} \mathbb{F}x^i y^{n-i},$$

we see that $M(n)$ is the direct sum of its weight spaces.

Lemma 2.8. *Suppose* $\mathbb{F}$ *is algebraically closed. Let* V *be a finite dimensional representation of* $\mathfrak{g}$. *Then the following claims hold:*

(1) *We have* $V = \bigoplus_{\lambda\in\mathbb{F}} V_\lambda$.
(2) *We have* $eV_\lambda \subset V_{\lambda+2}$ *and* $fV_\lambda \subset V_{\lambda-2}$.

Proof. (1) is standard. To prove (2), pick $v \in V_\lambda$. It follows that there is $\ell > 0$ with $(h-\lambda)^\ell v = 0$. By (1), $e(h-\lambda)^\ell = (h-\lambda-2)^\ell e$. So $(h-\lambda-2)^\ell ev = 0$, hence $ev \in V_{\lambda+2}$. Similarly, thanks to (2), we have $fV_\lambda \subset V_{\lambda-2}$. □

2.4. **Highest weight.** Until the end of Section 2.6, assume that $\mathrm{char}\,\mathbb{F} = 0$ and $\mathbb{F}$ is algebraically closed. Define a partial order on $\mathbb{F}$ by $\lambda \leqslant \mu$ if $\mu - \lambda \in 2\mathbb{Z}_{\geqslant 0}$. This gives an order on weights because $\mathrm{char}\,\mathbb{F} = 0$ implies that the natural map $\mathbb{Z} \to \mathbb{F}$ is an embedding.

Let V be a finite dimensional representation of $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$.

Definition 2.9. A weight λ of V is called a *highest weight* if λ is maximal with respect to this order. Any nonzero element in the corresponding subspace V_λ is called a *highest weight vector.*

Note that since $\dim V < \infty$, the set of weights of V is finite, so there exists a highest weight.

Example 2.10. By Example 2.7, n is the unique highest weight of M_n.

Proposition 2.11. *Let λ be a highest weight of V, and let $v \in V_\lambda$ be a nonzero vector. Then:*

(1) $ev = 0$,
(2) $\lambda \in \mathbb{Z}_{\geqslant 0}$ *and* $hv = \lambda v$.

Proof. By Lemma 2.8, $eV_\lambda \subset V_{\lambda+2}$. Since λ is the highest weight, there is no weight μ with $\mu > \lambda$, so $ev = 0$. This proves (1).

To prove (2) observe that there is $n > 0$ such that $\lambda - 2n$ is not a weight of V. Applying Lemma 2.8 again, we see that $f^n v = 0$. Combining the equalities $f^n v = 0, ev = 0$ with (5), we see that $[\prod_{i=0}^{n-1}(h-i)]v = 0$. We conclude that $\lambda \in \{0, \dots, n-1\}$ and $hv = \lambda v$. □

2.5. **Verma modules.** Proposition 1.4 implies that in every V there is a nonzero vector v such that $ev = 0$ and $hv = \lambda v$, where λ is a highest weight. We want to construct a universal module with such a vector, which, however, is not going to be finite dimensional.

Definition 2.12. Let $\lambda \in \mathbb{F}$. The *Verma module* $\Delta(\lambda)$ is defined as the quotient

$$\Delta(\lambda) = U(\mathfrak{g})/I,$$

where $I = U(\mathfrak{g})(h - \lambda, e)$ is the left ideal generated by the elements $h - \lambda$ and e.

The following proposition describes some properties of $\Delta(\lambda)$. Let $\bar{1}$ denote the image of 1 in $\Delta(\lambda)$.

Proposition 2.13. *The following claims hold.*

(1) *(Universal property) For any $\mathfrak{g}$-module V and any vector $v \in V$ with $ev = 0$ and $hv = \lambda v$, there exists a unique homomorphism $\varphi : \Delta(\lambda) \to V$ such that $\varphi(\bar{1}) = v$.*
(2) *(Basis) If $\lambda \in \mathbb{F}$ is arbitrary, then the vectors $f^i\bar{1}$ (for $i \geqslant 0$) form a basis in $\Delta(\lambda)$.*
(3) *(Submodules) $\Delta(\lambda)$ is simple if $\lambda \notin \mathbb{Z}_{\geqslant 0}$. If $\lambda \in \mathbb{Z}_{\geqslant 0}$, then $\Delta(\lambda)$ has a unique proper submodule. This submodule is the span of $f^n\bar{1}$ with $n > \lambda$.*

Proof. (1): the homomorphism is $\varphi_v : \Delta(\lambda) \to V, x + I \mapsto xv$. To prove its existence and uniqueness is left as an exercise.

(2): by the PBW theorem, the elements $f^i h^j e^k$ with $i, j, k \in \mathbb{Z}_{\geqslant 0}$ form a basis in $U(\mathfrak{g})$. The left ideal I is spanned by elements of the form $u(h - \lambda)$ and ue, so the elements $f^i\bar{1}$ form a basis in $\Delta(\lambda)$. This proves (2).

For (3), note that the elements $f^i\bar{1} \in \Delta(\lambda)$ form an eigenbasis for h with pairwise distinct eigenvalues $\lambda - 2i$ (with $i \in \mathbb{Z}_{\geqslant 0}$). So any submodule $N \subset \Delta(\lambda)$ is the span of some of these vectors, by an argument involving a Vandermonde determinant and left as an exercise. If $f^j\bar{1} \in N$, then $N \supset \mathrm{Span}\{f^i\bar{1} | i > j\}$. Now take the minimal element j such that $f^j\bar{1} \in N$. Since $ef^j\bar{1}$ is proportional to $f^{j-1}\bar{1}$, we see that $ef^j\bar{1}$. By (4), $ef^j = f^j e + jf^{j-1}(h - (j-1))$, so $ef^j\bar{1} = j(\lambda + 1 - j)f^{j-1}\bar{1}$. In particular, we see that if $\lambda \notin \mathbb{Z}_{\geqslant 0}$, then $\Delta(\lambda)$ is irreducible. To show that $\mathrm{Span}_{\mathbb{F}}(f^n\bar{1} | n > \lambda)$ is the unique proper submodule of $\Delta(\lambda)$ is left as an exercise. □

Thanks to (3), $\Delta(\lambda)$ has the unique irreducible quotient to be denoted by $L(\lambda)$. It coincides with $\Delta(\lambda)$ if $\lambda \notin \mathbb{Z}_{\geqslant 0}$ and has dimension $\lambda+1$ else. The proof of the following corollary is left as an exercise.

Corollary 2.14. *Let $\lambda \in \mathbb{F}, V$ be a finite dimensional representation of $\mathfrak{g}$, and $\varphi : \Delta(\lambda) \to V$ be a nonzero homomorphism. Then $\lambda \in \mathbb{Z}_{\geqslant 0}$, and φ factors as the composition of the projection $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ and the inclusion $L(\lambda) \hookrightarrow V$.*

2.6. **Completion of the classification of irreducibles.**

Proof of (1) of Theorem 2.4. First, we deal with the representations of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. The modules $L(n)$ introduced after the proof of Proposition 2.13 are pairwise non-isomorphic because $\dim L(n) = n+1$. To show that every irreducible $\mathfrak{sl}_2$-representation V is isomorphic to one of the modules $L(n)$ is an easy consequence of Corollary 2.14.

It remains to establish an isomorphism $M(n) \cong L(n)$ for $n \in \mathbb{Z}_{\geqslant 0}$. This is because $x^n \in M(n)$ is a highest weight vector that generates $M(n)$ as a $U(\mathfrak{g})$-module, the latter follows from Exercise 2.2.

Now we proceed to proving (1) for the rational representations of $G := \mathrm{SL}_2(\mathbb{F})$. By the previous paragraph, every irreducible representation of $\mathfrak{g}$ is of the form $M(n)$ for some $n \geqslant 0$, hence comes from a rational representation of $\mathrm{SL}_2(\mathbb{F})$. Hence there is an $\mathfrak{g}$-linear embedding $M(n) \to V$ for some n. By [2, Theorem 2.35], this embedding is also G-linear, showing that $V \cong M(n)$ and finishing the proof. □

Let us record a corollary of the proof.

Corollary 2.15. *We have an isomorphism $L(n) \cong M(n)$ of representations of $\mathfrak{g}$.*

Remark 2.16. (1) of Theorem 2.4 implies that the eigenvalues of h in any finite dimensional representation of $\mathfrak{g}$ (over an algebraically closed characteristic 0 field) are integers. Now consider the case when $\mathbb{F}$ is not necessarily algebraically closed but still has characteristic 0. It follows that any finite dimensional representation of $\mathfrak{sl}_2(\mathbb{F})$ splits as the direct sum of its weight spaces with integral eigenvalues. We can repeat the constructions in the proof of (1) of Theorem 2.4 and see that the direct analog of (1) holds over $\mathbb{F}$.

2.7. **Casimir element.** Our next task is to show that all finite dimensional representations of $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$ and all rational representations of $G := \mathrm{SL}_2(\mathbb{F})$ are completely reducible provided $\mathbb{F}$ is an algebraically closed field of characteristic 0 (in fact, similarly to Remark 2.16 the assumption that $\mathbb{F}$ is algebraically closed can be omitted). A new ingredient is a certain element in the center of $U(\mathfrak{g})$.

Here is a fundamental observation:

Proposition 2.17. *Suppose that $\mathbb{F}$ is a field of characteristic different from 2. Then the element $C := 2fe + \frac{1}{2}h^2 + h$ (known as the* Casimir element) is central.

Proof. $U(\mathfrak{g})$ is generated by e, h, f, so it's enough to check that $[C, e] = 0, [C, h] = 0, [C, f] = 0$. This is left as an exercise. □

In fact, one can understand this element more conceptually. We will do this in a later lecture. In the subsequent sections we will use C to prove the complete reducibility.

2.8. **Generalized eigenspaces.** From now on and until the further notice we assume that $\mathbb{F}$ is algebraically closed of characteristic 0. Let V be a finite-dimensional representation of $\mathfrak{g}$. For $z \in \mathbb{F}$, let V^z be the generalized eigenspace for C with eigenvalue z, i.e., V^z consists of all $v \in V$ such that $(C-z)^n v = 0$ for some n. Then we have $V = \bigoplus_z V^z$. The following proposition describes some properties of this decomposition.

Proposition 2.18. (1) *All V^z are $U(\mathfrak{g})$-submodules.*
(2) *If $V^z \neq \{0\}$, then $z = \frac{1}{2}n^2 + n$ for some $n \in \mathbb{Z}$. Moreover, $M(n)$ is the only irreducible constituent of V^z for such z.*

Proof. (1) holds because C is central. Details are left as an exercise.

We proceed to proving (2). Recall that the finite dimensional irreducible representations of $\mathfrak{g}$ are exactly the representations $M(n)$, (1) of Theorem 2.4. For a highest weight vector $v \in M(n)$ we have $ev = 0, hv = nv$. It follows that $Cv = (\frac{1}{2}n^2 + n)v$. On the other hand, by the Schur lemma, any central element of $U(\mathfrak{g})$ acts by a scalar on any irreducible finite dimensional module. It follows that C acts by the scalar $\frac{1}{2}n^2 + n$ on $M(n)$.

Now suppose $V^z \neq \{0\}$. Let $U_1 \subset U_2$ be two $\mathfrak{g}$-subrepresentations such that $U_2/U_1 \cong M(n)$ for some n. Then $CU_i \subset U_i$ for both $i = 1, 2$. It follows that C acts on $M(n)$ with generalized eigenvalue z. Hence $z = \frac{1}{2}n^2 + n$. And if $M(m)$ is another composition factor of V^z, then $\frac{1}{2}n^2 + n = \frac{1}{2}m^2 + m$. From here we easily deduce that $m = n$. □

2.9. **Complete Reducibility.**

Proof of (2) of Theorem 2.4. Consider the setting of the representations of $\mathfrak{g}$ first. Thanks to the decomposition $V = \bigoplus_z V^z$ and Proposition 2.18, we reduce to proving that $V^z \cong M(n)^{\oplus m}$ for some m if $z = \frac{1}{2}n^2 + n$. To simplify the notation we fix n and assume $V = V^z$ for $z = \frac{1}{2}n^2 + n$.

By (2) of Proposition 2.18, the $\mathfrak{g}$-module V admits a module filtration

$$0 = V^{(0)} \subseteq V^{(1)} \subseteq \cdots \subseteq V^{(m)} = V,$$

where $V^{(i+1)}/V^{(i)} \cong M(n)$ for all i.

Thanks to this filtration we see that n is the unique highest weight of V. If $V_n = V_n^{(m-1)}$, then the weights in $V/V^{(m-1)}$ are all less than n, a contradiction with $V/V^{(m-1)} \cong M(n)$. So pick a vector $v \in V_n \setminus V_n^{(m-1)}$. By Proposition 2.11, we have $ev = 0, hv = nv$. Consider the unique homomorphism $\varphi_v : \Delta(\lambda) \to V$ with $\varphi_v(\bar{1}) = v$, see (1) of Proposition 2.13. By Corollary 2.14, the image of φ_v is isomorphic to $L(n)$ (that is isomorphic to $M(n)$ by Corollary 2.15). Since $V/V^{(m-1)} \cong M(n)$ and $\operatorname{im}\varphi_v \not\subset V^{(m-1)}$, we conclude that $V = V^{(m-1)} \oplus M(n)$. Now we can apply the descending induction on m to finish the proof of $V \cong M(n)^{\oplus m}$.

Now we consider the setting of the rational representations of G. Let V be a rational representation. We can view V as a representation of $\mathfrak{g}$. By what we proved already, we get a $\mathfrak{g}$-linear isomorphism $V \cong M(n)^{\oplus m}$ for some m. But the right-hand side is a also rational representation of G. Using [2, Theorem 2.35] we see that the $\mathfrak{g}$-linear isomorphism $V \cong M(n)^{\oplus m}$ is, in fact, G-linear. This finishes the proof. □

Exercise 2.19. Prove a direct analog of (2) of Theorem 2.4 for finite dimensional representations of $\mathfrak{sl}_2$ over an arbitrary characteristic 0 field.

Remark 2.20. Informally speaking, the two main techniques going into the proof of Theorem 2.4 are studying highest weights of representations and the decomposition into a direct sum coming from a central element of $U(\mathfrak{g})$. In subsequent lectures we will see that these techniques are used to study various kinds of representations of semisimple algebraic group and their Lie algebras.

3. Representations of $\mathfrak{sl}_2$ in characteristic p

In this entire section we assume that $\mathbb{F}$ is an algebraically closed field of characteristic $p > 2$. Set $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. Our goal in this section is to classify the irreducible finite dimensional representations of $\mathfrak{g}$. We would like to emphasize that much of what we did in Section 2 no longer works. Most notably the notion of the order we used in Section 2.4 no longer makes sense because the natural homomorphism $\mathbb{Z} \to \mathbb{F}$ is not injective. As a consequence, we can no longer talk about highest weights.

3.1. **More central elements.** On the other hand, the element $C = 2fe + \frac{1}{2}h^2 + h \in U(\mathfrak{g})$ still makes sense and is central. It turns out that there are many more central elements in $U(\mathfrak{g})$.

Lemma 3.1. *The elements $e^p, f^p, h^p - h \in U(\mathfrak{g})$ are central.*

Proof. We need to show that these elements commute with the generators e, h, f of $U(\mathfrak{g})$. This follows from the formulas in Lemma 2.1. For example, f^p manifestly commutes with f, commutes with h by by (2), and commutes with e by (4). The rest of the check is left as an exercise. □

Definition 3.2. The central subalgebra of $U(\mathfrak{g})$ generated by the elements e^p, f^p and $h^p - h$ is called the *p-center* of $U(\mathfrak{g})$. We denote it by Z_p.

Here is an important property of the p-center. For the discussion below we assume that the PBW theorem, [2, Theorem 3.4], holds (we only proved it in the case of characteristic 0 fields).

Proposition 3.3. *The following claims hold:*

1. *The algebra Z_p is the polynomial algebra with free generators $f^p, h^p - h, e^p$.*
2. *The algebra $U(\mathfrak{g})$ is a free rank p^3 module over Z_p. Moreover, the elements $f^a h^b e^c$ with $0 \leqslant a, b, c \leqslant p-1$ form a basis of this module.*

Proof. We prove (2) leaving (1) as an exercise. By the PBW theorem, [2, Theorem 3.4], the elements $f^i h^j e^k$ with $i, j, k \in \mathbb{Z}_{\geqslant 0}$ form a basis of $U(\mathfrak{g})$ over $\mathbb{F}$. It follows that the elements of the form $f^a(f^p)^{i'} h^b (h^p - h)^{j'} e^c (e^p)^{k'}$ with $0 \leqslant a, b, c \leqslant p-1$ and $i', j', k' \in \mathbb{Z}_{\geqslant 0}$ also form a basis. From this, and an observation that $e^p, h^p - h, f^p$ are central, we arrive at the statement of the lemma. □

Consider an element $\chi := (\chi_f, \chi_h, \chi_e) \in \mathbb{F}^3$. We can form the quotient

$$U^\chi(\mathfrak{g}) := U(\mathfrak{g})/U(\mathfrak{g})(f^p - \chi_f, h^p - h - \chi_h, e^p - \chi_e) \tag{6}$$

of $U(\mathfrak{g})$. Since the elements $e^p, h^p - h, f^p$ are central, $U^\chi(\mathfrak{g})$ is an algebra, usually called a *p-central reduction of* $U(\mathfrak{g})$. We have the following corollary of Proposition 3.3.

Corollary 3.4. *We have* $\dim U^\chi(\mathfrak{g}) = p^3$ *for all* $\chi \in \mathbb{F}^3$. *In particular, for all* $\chi = (\chi_f, \chi_h, \chi_e)$ *there is a finite dimensional irreducible representation of* $\mathfrak{g}$, *where* f^p *acts by* χ_f, $h^p - h$ *acts by* χ_h, *and* e^p *acts by* χ_e.

Proof. The first claim is an immediate corollary of Lemma 3.3. The second claim follows by considering an irreducible constituent of the nonzero $\mathfrak{g}$-module $U^\chi(\mathfrak{g})$. □

In particular, we see that we cannot parameterize the finite dimensional irreducible representations of $\mathfrak{sl}_2$ in characteristic p by a discrete collection of parameters, a stark contrast with (1) of Theorem 2.4. However, in another aspect the representation theory in characteristic p is easier than in characteristic 0, as the following corollary of Proposition 3.3 shows. Recall that over a characteristic 0 field there are plenty of infinite dimensional irreducible representations of $\mathfrak{sl}_2$, e.g., the modules $\Delta(\lambda)$ with $\lambda \not\in \mathbb{Z}_{\geqslant 0}$ are irreducible, see Proposition 2.13.

Corollary 3.5. *Every irreducible $U(\mathfrak{g})$-module is finite dimensional.*

Proof. Let M be an irreducible $U(\mathfrak{g})$-module. It is generated by any of its nonzero elements, m. Then the elements $f^a h^b e^c m$ with $a, b, c \in \{0, \ldots, p-1\}$ generate M as a Z_p-module. The $\mathbb{F}$-algebra Z_p is a finitely generated domain by (1) of Proposition 3.3. From Commutative algebra (e.g., the Krull intersection theorem, [E, Corollary 5.4]) we know that there is a maximal ideal $\mathfrak{m} \subset Z_p$ such that $\mathfrak{m}M \neq M$. This maximal ideal must be of the form $(f^p - \chi_f, h^p - h - \chi_h, e^p - \chi_e)$ for some $\chi \in \mathbb{F}^3$. Since Z_p is contained in the center, $\mathfrak{m}M$ is a $U(\mathfrak{g})$-submodule. Since M is irreducible, $\mathfrak{m}M = \{0\}$. It follows that the elements $f^a h^b e^c m$ span M as an $\mathbb{F}$-vector space. This finishes the proof. □

3.2. **Classification of some irreducibles.** By the Schur lemma the central elements of $U(\mathfrak{g})$ act by scalars on every irreducible (finite dimensional) $U(\mathfrak{g})$-module M. In particular, to M we can assign its *p-character*, the triple $(\chi(f), \chi(h), \chi(e)) \in \mathbb{F}^3$ by which the elements $f^p, h^p - h, e^p$ act. The irreducible representations with a given p-character χ are the same thing as the irreducible representations of the algebra $U^\chi(\mathfrak{g})$ defined by (6). So, it is enough to classify the irreducible representations of $U^\chi(\mathfrak{g})$ for each χ. In this section we will handle three special cases (two individual triples and one family). In the subsequent sections we will reduce the classification to these special cases by analyzing the p-center in a more detailed and conceptual fashion.

The cases we consider are:

- $\chi = 0$,
- $\chi := \chi^0$ given by $\chi^0(e) = \chi^0(h) = 0, \chi^0(f) = 1$,
- $\chi := \chi^\alpha$ given $\chi^\alpha(e) = \chi^\alpha(f) = 0, \chi^\alpha(h) := \alpha \in \mathbb{F} \setminus \{0\}$.

Case 1: $\chi = 0$. We begin with an exercise that is an easy consequence of Exercise 2.2.

Exercise 3.6. The elements $e^p, f^p, h^p - h$ act by 0 on all representations $M(n)$.

Lemma 3.7. *The irreducible $U^0(\mathfrak{g})$-modules are exactly $M(m)$ for $m = 0, \ldots, p-1$.*

Proof. The proof is in several steps. Let V be a finite dimensional $U^0(\mathfrak{g})$-module.

Step 1. $h^p - h$ acts by 0 on V. Note the equality $x^p - x = \prod_{i=0}^{p-1}(x - i)$ in $\mathbb{F}_p[x]$. So $V = \bigoplus_{\lambda \in \mathbb{F}_p} V_\lambda$, where V_λ is the eigenspace for h with eigenvalue λ. Since $e^p = 0$, we have $\ker e \neq 0$. And since $eV_\lambda \subseteq V_{\lambda+2}$, we have $\ker e = \bigoplus_{i=0}^{p-1}(V_\lambda \cap \ker e)$. It follows that there is a nonzero element $v \in V$ with $hv = \lambda v, ev = 0$ for some $\lambda \in \mathbb{F}_p$.

Step 2. The Verma module $\Delta(\lambda) := U(\mathfrak{g})/U(\mathfrak{g})(h - \lambda, e)$ still makes in this setting. It has the same properties as in characteristic 0, Proposition 2.13, the proof carries to characteristic p verbatim. In particular, we have a unique homomorphism $\varphi : \Delta(\lambda) \to V$ with $\bar{1} \mapsto v$. Since f^p is central, the subspace $f^p\Delta(\lambda) \subset \Delta(\lambda)$ is a submodule. Since f^p acts on V by 0, the homomorphism φ factors through the quotient $\underline{\Delta}^0(\lambda) := \Delta(\lambda)/f^p\Delta(\lambda)$ (known as the *baby Verma module*).

Step 3. Let v_λ be the image of $\bar{1}$ in $\underline{\Delta}^0(\lambda)$. Then the elements

$$f^j v_\lambda, \quad 0 \leqslant j < p,$$

form a basis of $\underline{\Delta}^0(\lambda)$. From here, we can analyze the submodules of $\underline{\Delta}^0(\lambda)$ similarly to the proof of Proposition 2.13. We find that $\underline{\Delta}^0(p-1)$ is irreducible, while for $i \in \{0, 1, \ldots, p-2\}$, the only proper submodule of $\underline{\Delta}^0(i)$ is $\mathrm{Span}_{\mathbb{F}}(f^{\lambda+1}v_\lambda, \ldots, f^{p-1}v_\lambda)$. The proof from now on essentially repeats the argument in Section 2.6. □

Remark 3.8. The module $\underline{\Delta}^0(\lambda)$ for $\lambda \neq p-1$ is not completely reducible.

Case 2: $\chi = \chi^0$.

Lemma 3.9. *The algebra $U^\chi(\mathfrak{g})$ has exactly $(p+1)/2$ pairwise nonisomorphic irreducible representations, all of dimension p.*

Proof. As in the proof of Lemma 3.7, for any finite dimensional $U^\chi(\mathfrak{g})$-module V we can find a nonzero vector v with $ev = 0, hv = \lambda v$ for some $\lambda \in \mathbb{F}_p$. Set $\underline{\Delta}^\chi(\lambda) = \Delta(\lambda)/(f^p - 1)\Delta(\lambda)$ and let v_λ denote the image of $\bar{1} \in \Delta(\lambda)$. Unlike in the case of $\chi = 0$, this module is irreducible for all λ (hint: look at the weight decomposition and use that f^p acts by 1). Any irreducible $U^\chi(\mathfrak{g})$-module must be isomorphic to one of $\underline{\Delta}^\chi(\lambda)$.

Moreover, for $\lambda \neq p-1$, the kernel of e in $\underline{\Delta}^\chi(\lambda)$ also contains the nonzero vector $f^{\lambda+1}v_\lambda$ of weight $-2-\lambda$. This yields an isomorphism $\underline{\Delta}^\chi(\lambda) \cong \underline{\Delta}^\chi(-2-\lambda)$.

Finally, suppose that we have an isomorphism $\underline{\Delta}^\chi(\lambda) \cong \underline{\Delta}^\chi(\lambda')$ for $\lambda, \lambda' \in \mathbb{F}_p$. The Casimir element C acts on $\underline{\Delta}^\chi(\lambda)$ by the scalar $\frac{\lambda^2}{2} + \lambda$. From here we deduce that $\lambda' = \lambda$ or $\lambda' = -2-\lambda$ finishing the proof. □

Case 3: $\chi = \chi^\alpha$ *for* $\alpha \neq 0$. Here we can still form the baby Verma modules $\underline{\Delta}^\chi(\lambda) = \Delta(\lambda)/f^p\Delta(\lambda)$, where λ is a solution of $x^p - x = \alpha$ (there are p different solutions). They are modules over $U^\chi(\mathfrak{g})$. The following claim can be checked similarly to the proof of Lemma 3.7.

Exercise 3.10. There are p pairwise non-isomorphic $U^\chi(\mathfrak{g})$-modules and these are exactly the modules $\underline{\Delta}^\chi(\lambda)$, where λ runs over the set of solutions of $x^p - x = \alpha$.

3.3. ***p*-center, revisited.** In this section we will obtain a more conceptual description of the p-center of $U(\mathfrak{sl}_2)$. We will work more generally with an algebraic subgroup $G \subset \mathrm{GL}_n(\mathbb{F})$. In particular, we can view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$. For a matrix $\xi \in \mathfrak{gl}_n(\mathbb{F})$ we write $\xi^{[p]}$ for its pth power (as a matrix; we will explain why this fancy notation is necessary below). The following result is analogous to the first part of [2, Theorem 2.18], we will prove this as well as an analog of part (2) of that theorem in Section 3.5, where some constructions of this section will be analyzed more conceptually.

Proposition 3.11. *We have $\xi^{[p]} \in \mathfrak{g}$ for all $\xi \in \mathfrak{g}$.*

Exercise 3.12. Check this explicitly for the subalgebras $\mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F})$ and $\mathfrak{sp}_n(\mathbb{F})$ (the latter is for even n).

The map $\xi \mapsto \xi^{[p]} : \mathfrak{g} \to \mathfrak{g}$ is called the *restricted pth power map.* We use the square brackets to distinguish this map from taking the pth power in the universal enveloping algebra $U(\mathfrak{g})$.

Define the map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ by

$$\iota(x) := x^p - x^{[p]}. \tag{7}$$

The following example connects ι to the p-center of $U(\mathfrak{sl}_2(\mathbb{F}))$.

Example 3.13. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$, we have $\iota(e) = e^p, \iota(f) = f^p, \iota(h) = h^p - h$.

Below in this section we establish some properties of ι that will be used to complete the classification of the irreducible representations of $\mathfrak{sl}_2(\mathbb{F})$.

Lemma 3.14. *Let G be as in the beginning of the section. Then the element $\iota(x) \in U(\mathfrak{g})$ is central for all $x \in \mathfrak{g}$.*

Proof. The proof is in two steps.

Step 1. Let A be an associative algebra over $\mathbb{F}$ and let $a \in A$. We can consider the linear map $\mathrm{ad}_a : A \to A, b \mapsto ab - ba$. We claim that

$$(\mathrm{ad}_a)^p = \mathrm{ad}_{a^p}\,. \tag{8}$$

Indeed, let $L_a, R_a : A \to A$ be the operators of the left and right multiplication by a so that $L_a(b) := ab, R_a(b) := ba$. Then $\mathrm{ad}_a = L_a - R_a$. The operators $L_a, R_a \in \mathrm{End}_{\mathbb{F}}(A)$ commute. It follows that

$$(L_a - R_a)^p = \sum_{i=0}^{p} (-1)^i \binom{p}{i} L_a^i R_a^{p-i} = [\binom{p}{i} = 0, \forall i = 1, \ldots, p-1] = L_a^p - R_a^p = \mathrm{ad}_{a^p}\,.$$

This shows (8).

Step 2. Applying (8) to $A = U(\mathfrak{g})$ we get $[x^p, y] = \mathrm{ad}_x^p(y)$ for all $x, y \in \mathfrak{g}$, note that the right hand side is an element of $\mathfrak{g}$. Similarly, applying (8) to $A = \mathrm{Mat}_n(\mathbb{F})$, we get $[x^{[p]}, y] = \mathrm{ad}_x^p\, y$. We conclude that $[\iota(x), y] = 0$ for all $x, y \in \mathfrak{g}$, hence $\iota(x)$ is indeed central. □

It is not obvious that in the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$ the elements $\iota(x)$ lie in the p-center for an arbitrary x. This actually follows from the next lemma.

Lemma 3.15. *Let $a \in \mathbb{F}, x, y \in \mathfrak{g}$. Then the following claims hold:*

(1) $\iota(ax) = a^p \iota(x)$,
(2) *and* $\iota(x + y) = \iota(x) + \iota(y)$.

Proof. (1) is straightforward and is left as an exercise. The proof of (2) is in several steps. Set $z := \iota(x + y) - \iota(x) - \iota(y)$.

Step 1. Recall the PBW filtration $U(\mathfrak{g})_{\leqslant i}, i \in \mathbb{Z}_{\geqslant 0}$, on $U(\mathfrak{g})$, see the discussion after Example 3.14 in [2, Sec. 3.3]. We write $\bar{x}, \bar{y}$ for the images of $x, y \in U(\mathfrak{g})_{\leqslant 1}$ in $\mathrm{gr}\, U(\mathfrak{g})$. Notice that $\mathrm{gr}\, U(\mathfrak{g})$ is a quotient of $S(\mathfrak{g})$, hence commutative. The element z belongs to $U(\mathfrak{g})_{\leqslant p}$ by the construction. Note however that its image in $U(\mathfrak{g})_{\leqslant p}/U(\mathfrak{g})_{\leqslant p-1}$ equals to $(\bar{x} + \bar{y})^p - \bar{x}^p - \bar{y}^p$. The latter element is 0 because $\mathrm{gr}\, U(\mathfrak{g})$ is commutative. It follows that

$$z \in U(\mathfrak{g})_{\leqslant p-1}. \tag{9}$$

Step 2. Recall, [2, Example 4.5], that $U(\mathfrak{g})$ has the unique Hopf algebra structure with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$. Note that

$$\Delta(x^p) = \Delta(x)^p = (x \otimes 1 + 1 \otimes x)^p = x^p \otimes 1 + 1 \otimes x^p,$$

the last equality holds because the elements $x \otimes 1, 1 \otimes x \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ commute. It follows that $\Delta(\iota(x)) = \iota(x) \otimes 1 + 1 \otimes \iota(x)$ for any $x \in \mathfrak{g}$. Hence

$$\Delta(z) = z \otimes 1 + 1 \otimes z. \tag{10}$$

Step 3. Combining (9) and (10) with [2, Exercise 4.14], we see that $z \in \mathfrak{g}$. Note that since z lies in the center of $U(\mathfrak{g})$, it must lie in the center of $\mathfrak{g}$ (the subspace of all elements that commute with every other element). If the center is $\{0\}$, then we are done. Otherwise, we can argue as follows. First, consider the embedding $\Phi : G \hookrightarrow \mathrm{SL}_m(\mathbb{F})$, where $m > n$ is coprime to p, given by $g \mapsto \mathrm{diag}(g, \det(g)^{-1}, 1, \dots, 1)$ (a block diagonal matrix, where we consider G as a subgroup in $\mathrm{GL}_n(\mathbb{F})$). Let $\varphi : \mathfrak{g} \to \mathfrak{sl}_m(\mathbb{F})$ denote the corresponding embedding of Lie algebras. It is easy to show that

(a) the center of $\mathfrak{sl}_m(\mathbb{F})$ is $\{0\}$,
(b) and $\varphi(x^{[p]}) = \varphi(x)^{[p]}$ for all $x \in \mathfrak{g}$.

These claims are left as exercises. Using the embedding φ we reduce the problem of showing $z = 0$ to the case when the center is $\{0\}$, which finishes the proof. □

Remark 3.16. Here is a stronger statement. Let $\mathfrak{fl}_2$ denote the free Lie algebra in two generators x, y. A direct analog of [2, Execise 4.14] holds for $\mathfrak{fl}_2$, for example, this follows from the PBW theorem. The universal enveloping algebra $U(\mathfrak{fl}_2)$ is identified with the free associative algebra $\mathbb{F}\langle x, y\rangle$ because they satisfy the same universal property. We see that $(x+y)^p - x^p - y^p \in \mathbb{F}\langle x, y\rangle$ actually lies in $\mathfrak{fl}_2$. Denote this element by $L_p(x, y)$. We conclude that $(a+b)^p - a^p - b^p = L_p(a, b)$ for every associative $\mathbb{F}$-algebra A and any elements $a, b \in A$. This gives a somewhat alternative proof of Lemma 3.15.

Finally, we need equivariance properties of the map ι. Recall the representation Ad of G in $\mathfrak{g}$, [2, Lemma 2.28]. Note that $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism for any $g \in G$. Indeed, by the construction, Ad_g is given by $x \mapsto gxg^{-1}$ when we view $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_n(\mathbb{F})$. This map is a Lie automorphism of $\mathfrak{gl}_n(\mathbb{F})$, hence of $\mathfrak{g}$ as well. Therefore, the adjoint action of G lifts to an action on $U(\mathfrak{g})$ by automorphisms because G acts on $T(\mathfrak{g})$ by algebra automorphisms and preserves the ideal $(x \otimes y - y \otimes x - [x, y] | x, y \in \mathfrak{g})$.

Exercise 3.17. The map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ intertwines the G-actions: $\iota(\mathrm{Ad}_g x) = \mathrm{Ad}_g \iota(x)$.

3.4. **Completion of classification.** Let V be a (finite dimensional) irreducible representation of $U(\mathfrak{g})$. Combining Lemma 3.14 with the Schur lemma, we see that $\iota(x)$ acts by a scalar, to be denoted by $\chi(x)$, for any $x \in \mathfrak{g}$. For $x = e, h, f$ we recover the scalars $\chi(e), \chi(h), \chi(f)$ from Section 3.2. By Lemma 3.15, $x \mapsto \chi(x)$ is a semi-linear function $\mathfrak{g} \to \mathbb{F}$, where "semi" refers to the identity $\chi(ax) = a^p\chi(x)$ for all $a \in \mathbb{F}, x \in \mathfrak{g}$. Thanks to the semilinearity, χ is uniquely recovered from the triple $(\chi(f), \chi(h), \chi(e))$, so we can view the p-character of V as a semilinear function $\mathfrak{g} \to \mathbb{F}$. Denote the space of such functions by $\mathfrak{g}^{*(1)}$. Note that $\mathfrak{g}^{*(1)}$ is naturally an $\mathbb{F}$-vector space.

In Section 3.2 we have classified the irreducible representations of $U^\chi(\mathfrak{g})$ for $\chi = 0, \chi^0$ and χ^a for $a \neq 0$. Below we will see that there is a representation of G in $\mathfrak{g}^{*(1)}$ such that the algebras $U^\chi(\mathfrak{g})$ and $U^{g\chi}(\mathfrak{g})$ are isomorphic. We will compute the representation of G in $\mathfrak{g}^{*(1)}$ explicitly and see that every G-orbit contains $0, \chi^0$ or χ^a with $a \neq 0$. This will complete the classification of irreducible $U(\mathfrak{g})$-modules.

To begin with, we observe that $(x, y) := \mathrm{tr}(xy) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is a nondegenerate symmetric bilinear form. It is invariant for the adjoint action of G and so gives rise to an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ of representations of G. Next, note that we have a natural nondegenerate sesquilinear pairing $\mathfrak{g}^{*(1)} \times \mathfrak{g} \to \mathbb{F}, \langle \varphi, x\rangle = \varphi(x)$. There is a unique representation of G in $\mathfrak{g}^{*(1)}$ making the pairing invariant. This representation can be computed as follows. Consider the map $\mathrm{Fr} : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix},$$

cf. [2, Example 1.10]. This is a semilinear isomorphism $\mathfrak{g} \to \mathfrak{g}$. We can identify $\mathfrak{g}^{*(1)}$ with $\mathfrak{g}$ by sending $x \in \mathfrak{g}$ to the semilinear function $y \mapsto \mathrm{tr}(x\,\mathrm{Fr}(y))$. Under this identification, the action of $\mathfrak{g}^{*(1)}$ introduced above becomes $g \mapsto \mathrm{Ad}_{\mathrm{Fr}(g)}$.

Next we establish a compatibility result between the G-actions on $\mathfrak{g}^{*(1)}$ and on $U(\mathfrak{g})$.

Lemma 3.18. *For any $g \in G$, the automorphism Ad_g of $U(\mathfrak{g})$ sends $\mathrm{Span}_{\mathbb{F}}(\iota(x) - \langle \chi, x\rangle | x \in \mathfrak{g})$ to $\mathrm{Span}_{\mathbb{F}}(\iota(x) - \langle g\chi, x\rangle | x \in \mathfrak{g})$.*

Proof. For example, let us show that $g(\iota(e) - \langle \chi, e\rangle)$ lies in $\mathrm{Span}_{\mathbb{F}}(\iota(x) - \langle g\chi, x\rangle | x \in \mathfrak{g})$. By Exercise 3.17, $g\iota(e) = \iota(ge)$. We can write ge as $ae + bh + cf$ for $a, b, c \in \mathbb{F}$. Then $\iota(ge) = a^p\iota(e) + b^p\iota(h) + c^p\iota(f)$. Similar computations for h, f imply the matrix of $\mathrm{Ad}(g)$ in the basis $\iota(e), \iota(h), \iota(f)$ of $\iota(\mathfrak{g})$ is obtained by applying Fr to the matrix of $\mathrm{Ad}(g)$ in the basis e, h, f of $\mathfrak{g}$. The result of the lemma follows from this observation and the descriptions of the G-actions on $\mathfrak{g}^*$ and $\mathfrak{g}^{*(1)}$ given before the lemma. □

Thanks Lemma 3.18, the automorphism Ad_g of $U(\mathfrak{g})$ sends $\ker[U(\mathfrak{g}) \to U^\chi(\mathfrak{g})]$ to $\ker[U(\mathfrak{g}) \to U^{g\chi}(\mathfrak{g})]$ and hence gives rise to an isomorphism of quotients of $U(\mathfrak{g})$:

$$\mathrm{Ad}_g : U^\chi(\mathfrak{g}) \xrightarrow{\sim} U^{g\chi}(\mathfrak{g}).$$

To complete the classification, we need to show that any element of $\mathfrak{g}^{*(1)}$ is conjugate under G to 0 or χ^α for $\alpha \in \mathbb{F}$. Since the G-action is linear we only need to treat the case of nonzero elements χ. Under the identification of $\mathfrak{g}^{*(1)}$ with $\mathfrak{g}$, the element χ^0 corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, while χ^α becomes $\begin{pmatrix} \alpha/2 & 0 \\ 0 & -\alpha/2 \end{pmatrix}$. The action of G is by $g \mapsto \mathrm{Ad}_{\mathrm{Fr}(g)}$. Since Fr is an automorphism of $\mathbb{F}$, the orbits for this action are the matrix conjugacy classes. And our claim in the beginning of the paragraph is a consequence of the Jordan normal form theorem.

3.5. **Restricted pth power map, conceptually.** The definition of the restricted pth power map given in Section 3.3 has several disadvantages. First, we have not proved Proposition 3.11, so it is not clear whether $x^{[p]}$ makes sense for a general algebraic group G. Second, even if $x^{[p]}$ makes sense, it is unclear whether it is independent of the embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{F})$. The following theorem, that should be compared to [2, Theorem 2.18], fixes the issues.

Theorem 3.19. *Let G be an algebraic group.*

(1) *The claim of Proposition 3.11 holds, i.e., for any embedding $\iota : G \to \mathrm{GL}_n(\mathbb{F})$ (and the induced embedding $\iota : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{F})$) we have $\iota(x)^p \in \iota(\mathfrak{g})$.*
(2) *Let $\Phi : G \to H$ be an algebraic group homomorphism and let $\varphi := d_1\Phi : \mathfrak{g} \to \mathfrak{h}$. Define the restricted pth power maps for $\mathfrak{g}$ and $\mathfrak{h}$ using some embeddings. Then $\varphi(x^{[p]}) = \varphi(x)^{[p]}$.*

Proof. The proof will be given by using the $\mathbb{F}[\epsilon]/(\epsilon^k)$-points approach outlined in [2, Sec. 2.4]. Let G_k denote the group of $\mathbb{F}[\epsilon]/(\epsilon^k)$-points of G.

Step 1. An important fact we have not mentioned so far is that the natural map $G_{k+1} \to G_k$ is surjective. This follows from two observations. First, G is smooth, see [2, Exercise 2.10]. The second observation is the infinitesimal lifting property, see Exercise 8.6 in [H, Chapter 2]. The details on how the required statement follows are left as an exercise. Alternatively, the readers are encouraged to check the surjectivity claim for the classical groups.

Step 2. Take an element g of the form $1 + \sum_{i=1}^p x_i\epsilon^i$ in $\mathrm{GL}_n(\mathbb{F})_{p+1} (= \mathrm{GL}_n(\mathbb{F}[\epsilon]/(\epsilon^{p+1})))$, where $x_i \in \mathfrak{gl}_n(\mathbb{F})$. Then $g^p = 1 + x_1^p\epsilon^p$.

Step 3. Now pick $\xi \in \mathfrak{g}$ and set $x_1 = \iota(\xi)$. By Step 1, we can find $x_2, \ldots, x_p \in \mathfrak{gl}_n(\mathbb{F})$ such that $g^p = 1 + \sum_{i=1}^p x_i \epsilon^p \in \iota(G_{p+1})$. Since $g^p \in \iota(G_{p+1})$ and the kernel of $\iota(G_{p+1}) \twoheadrightarrow \iota(G_p)$ coincides with $\iota(\mathfrak{g})$, cf. [2, Sec. 2.4], we deduce part (1) of the theorem.

Step 4. Part (2) is proved similarly, this is left as an exercise (cf. Step 5 in the proof of Theorem 2.18 in [2, Sec. 2.3]). □

4. Representations of $\mathrm{SL}_2(\mathbb{F})$ with $\mathrm{char}\,\mathbb{F} > 2$

Our task in this section is to classify the irreducible rational representations of $G := \mathrm{SL}_2(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic $p > 2$.

4.1. **Construction of irreducibles.** The representations $M(\lambda)$ from Section 2.2, i.e., the space of homogeneous polynomials in x, y of degree λ with $\lambda \in \mathbb{Z}_{\geqslant 0}$, still make sense but generally are no longer irreducible.

Example 4.1. The representation $M(\lambda)$ is irreducible over the Lie algebra $\mathfrak{g}$ when $\lambda = 0, 1, \ldots, p-1$, see Lemma 3.7. And since every G-subrepresentation is also a $\mathfrak{g}$-subrepresentation, $M(\lambda)$ is irreducible over G. However, the G-representation $M(p)$ is not irreducible, indeed $\mathrm{Span}(x^p, y^p) \subset M(p)$ is is a G-subrepresentation. In fact, the representations $M(n)$ for $n \geqslant p$ are not completely reducible.

To produce more irreducible objects, we introduce the construction called the "Frobenius twist".

Definition 4.2. Let V be a rational representation of G, and let $\rho : G \to \mathrm{GL}(V)$ be the corresponding homomorphism. The *Frobenius twist* $V^{(1)}$ is the representation corresponding to the homomorphism $\rho^{(1)} : G \to \mathrm{GL}(V)$ given by $\rho^{(1)} := \rho \circ \mathrm{Fr}$, where $\mathrm{Fr} : G \to G$ is the Frobenius homomorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$.

Exercise 4.3. The subrepresentation $\mathrm{Span}(x^p, y^p) \subset M(p)$ is isomorphic to $M(1)^{(1)}$.

Notice that since Fr is an abstract group isomorphism, we have

(*) $V^{(1)}$ is irreducible if and only if V is.

In fact, we can construct a lot of irreducible representations of G using the Frobenius twists and tensor products. The following proposition generalizing (*) provides an inductive tool to do this.

Proposition 4.4. *If V is an irreducible rational representation of G, then $M(\lambda) \otimes V^{(1)}$ is irreducible for all $\lambda = 0, 1, \ldots, p-1$.*

Proof. If U is a nonzero G-subrepresentation of $M(\lambda) \otimes V^{(1)}$, then it is also a $\mathfrak{g}$-subrepresentation. Recall, Lemma 3.7, that $M(\lambda)$ is an irreducible representation of $\mathfrak{g}$. By [1, Proposition 2.17], $U = M(\lambda) \otimes U_0$, where $U_0 \subset V^{(1)}$ is a subspace. In order for U to be a G-subrepresentation, U_0 must be a G-subrepresentation. But $V^{(1)}$ is irreducible, so we must have $U_0 = V^{(1)}$ proving the claim. □

The proposition gives a way to produce irreducible representations. Namely, for a rational representation V of G we define representations $V^{(k)}$ for $k > 1$ inductively by $V^{(k)} = (V^{(k-1)})^{(1)}$.

Corollary 4.5. *Let $\lambda_0, \ldots, \lambda_k \in \{0, 1, 2, \ldots, p-1\}$. Then the representation*

$$\bigotimes_{i=0}^{k} M(\lambda_i)^{(i)} \tag{11}$$

is irreducible.

Here is the main result of Section 4.

Theorem 4.6. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 2$. Every irreducible rational representation of $\mathrm{SL}_2(\mathbb{F})$ is isomorphic to an irreducible representation of the form (11) for uniquely determined $k, \lambda_0, \ldots, \lambda_k$.*

The proof will be given below in this section after some preparation.

4.2. **Weight decomposition.** Our first step in proving Theorem 4.6 is to establish an analog of of the weight decomposition, see Section 2.3.

Inside $G := \mathrm{SL}_2(\mathbb{F})$ consider the subgroup T of all diagonal matrices, i.e.,

$$T = \{\mathrm{diag}(z, z^{-1}) | z \in \mathbb{F}^\times\}.$$

Projecting to the first entry gives rise to an isomorphism $T \xrightarrow{\sim} \mathbb{G}_m$.

Lemma 4.7. *The following claims hold:*

1. *Any rational representation V of T splits into the direct sum of (automatically rational) one dimensional representations of T.*
2. *Any 1-dimensional rational representation of T is isomorphic to the representation in $\mathbb{F}$ given by $z \mapsto z^m$ for a unique integer m.*

Proof. We prove (1). From Linear Algebra, we know that any two commuting diagonalizable operators are simultaneously diagonalizable. In fact, this is true for any collection (even infinite) of diagonalizable operators, this is left as an exercise. The collection we take consists of all elements of $\mathbb{F}^\times$ that are of finite order coprime to p hence diagonalizable, denote it by $\mathcal{C}$. So the elements from $\mathcal{C}$ are simultaneously diagonalizable.

Take the corresponding basis and consider the matrix coefficients of the representation in this basis. The non-diagonal matrix coefficients vanish on $\mathcal{C}$ and so are zero. Notice that $\mathcal{C}$ is Zariski dense in $\mathbb{G}_m$, so the nondiagonal matrix coefficients are zero. This proves (1).

Now we prove (2). To give a 1-dimensional rational representation of T amounts to giving an element $f \in \mathbb{F}[\mathbb{G}_m] = \mathbb{F}[z^{\pm 1}]$ such that $f(z_1 z_2) = f(z_1) f(z_2)$, an equality of elements in $\mathbb{F}[z_1^{\pm 1}, z_2^{\pm 1}]$. From here it is easy to deduce that $f(z) = z^m$ for some $m \in \mathbb{Z}$. □

We get back to the situation when V is a finite dimensional rational representation of G. Thanks to Lemma 4.7, we can decompose V into the direct sum

$$V = \bigoplus_{n \in \mathbb{Z}} V_n, \text{ where } V_n := \{v \in V | \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} v = z^n v, \forall z \in \mathbb{G}_m\}$$

Definition 4.8. By a highest (resp., lowest) weight of V we mean the maximal (resp., minimal) n such that $V_n \neq \{0\}$.

Example 4.9. The weights of $M(n)$ are $n, n-2, n-4, \ldots, -n$. In particular, the highest weight is n and the lowest weight is $-n$.

Exercise 4.10. Show that the action of the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$ restricts to an isomorphism $V_n \xrightarrow{\sim} V_{-n}$ for all $n \in \mathbb{Z}$. In particular, the highest and the lowest weights of V are opposite, and the highest weight is a nonnegative integer.

We are going to deduce Theorem 4.6 from the following proposition that will be proved in a subsequent section. The result is similar in spirit to the proof of Theorem 2.4 given in Section 2.6.

Proposition 4.11. *An irreducible rational representation of G is uniquely determined by its highest weight.*

Proof of Theorem 4.6 modulo Proposition 4.11. Theorem boils down to showing that the highest weight of (11) is $\sum_{i=0}^{k} \lambda_i p^i$ (indeed, every positive integer admits the unique p-adic expression). This claim in turn boils down to the following two easy observations that are left as exercises.

- For a rational finite dimensional representation V of G, we have $V_n^{(1)} = \{0\}$ if n is not divisible by p and $V_n^{(1)} = V_{n/p}$ else.
- For rational finite dimensional representations U and V of G, we have $(U \otimes V)_n = \bigoplus_{m \in \mathbb{Z}} U_m \otimes V_{n-m}$.

□

4.3. **Induced Modules.** The main ingredient in proving Proposition 4.11 is the claim that every irreducible representation V with lowest weight $-\lambda$ embeds into $M(\lambda)$ (this is morally similar to the claim that any finite dimensional representation of $\mathfrak{sl}_2$ in characteristic 0 is a quotient of the Verma module $\Delta(\lambda)$, the fact established in Section 2.6; we will further comment on this analogy in Remark 4.15). For this, we need to realize $M(\lambda)$ as an *induced representation.* Recall that the induced representations in the context of finite groups were introduced in [1, Section 3.5].

Now let $H \subset G$ be algebraic groups, and U be a rational finite dimensional representation of H. Note that both G and U are algebraic varieties so we can consider morphisms $G \to U$.

Definition 4.12. The *algebraic induced representation* is

$$\mathrm{Ind}_H^G U = \{\text{morphisms } f : G \to U \mid f(hg) = hf(g) \, \forall h \in H, g \in G\}, \tag{12}$$

where the structure of an $\mathbb{F}$-vector space is by pointwise operations (say, $[f_1 + f_2](g) = f_1(g) + f_2(g)$) and G acts on $\mathrm{Ind}_H^G U$ by $[g.f](g') = f(g'g)$ for $f \in \mathrm{Ind}_H^G U$ and $g, g' \in G$.

In general, $\mathrm{Ind}_H^G U$ is infinite dimensional (the easiest example is when $H = \{1\}$ and G is infinite), but one can still show that it is equal to the sum of its finite dimensional rational subrepresentations. We are not going to do this.

The following claim is an algebraic group analog of the second isomorphism in [1, Corollary 3.13]. We would like to point out, however, that the first isomorphism there does not hold in the algebraic context.

Lemma 4.13. *The Frobenius reciprocity holds: for any finite dimensional rational representations V of G and U of H we have a natural isomorphism*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U) \cong \mathrm{Hom}_H(V, U).$$

Proof. Let $v \mapsto \varphi_v$ be an element of $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U)$. For any $v \in V$, we have that φ_v is a morphism $G \to U$, so that we can evaluate it at $1 \in G$. Now $v \mapsto \varphi_v(1)$ is a linear map $V \to U$.

Let us show that $v \mapsto \varphi_v(1)$ is H-equivariant. By the H-equivariance condition in (12) applied to $g = 1$, we have $\varphi_v(h) = h(\varphi_v(1))$. On the other hand, by the definition of the G-action on $\mathrm{Ind}_H^G U$, we have $\varphi_v(h) = [h\varphi_v](1)$. Since φ is G- (and, in particular, H-) equivariant, we have $[h\varphi_v] = \varphi_{hv}$. Combining the equations in this paragraph, we arrive at

$$h(\varphi_v(1)) = \varphi_v(h) = [h\varphi_v](1) = \varphi_{hv}(1),$$

showing that the map $v \mapsto \varphi_v(1)$ is indeed H-equivariant. This gives rise to a linear map

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U) \to \mathrm{Hom}_H(V, U).$$

Now take $\psi \in \mathrm{Hom}_H(V, U)$. We want to construct a linear map $\Phi_\psi : V \to \mathrm{Ind}_H^G U$. Take $v \in V$ and let $\Phi_\psi(v)$ denote the following map: $g \mapsto \psi(gv)$. It is easy to see that $\Phi_\psi(v) : G \to U$ is a morphism (because the action map $G \times V \to V$ is a morphism). Also $\Phi_\psi(v) \in \mathrm{Ind}_H^G U$: this is because $[\Phi_\psi(v)](hg) = \psi(hgv) = h\psi(gv) = [h(\Phi_\psi(v))](g)$. Similarly, one checks that Φ_ψ is G-equivariant. This gives rise to a linear map

$$\mathrm{Hom}_H(V, U) \to \mathrm{Hom}_G(V, \mathrm{Ind}_H^G U).$$

To show that the maps in the two previous paragraphs are mutually inverse is left as an exercise. □

Now we explain how to construct $M(\lambda)$ as an induced representation. Consider the subgroup $B \subset G$ consisting of all upper triangular matrices, i.e., $B = \{\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}\}$. For $m \in \mathbb{Z}$, we write $\mathbb{F}_m$ for the 1-dimensional representation, where $\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}$ acts by t^m.

Proposition 4.14. *We have an isomorphism of G-representations*

$$M(\lambda) \cong \mathrm{Ind}_B^G \mathbb{F}_{-\lambda}.$$

Proof. Note that the right hand side is nothing else but the space of all polynomial functions f on G satisfying

$$f(\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix} g) = t^{-\lambda} f(g), \forall t \in \mathbb{F}^\times, z \in \mathbb{F}, g \in G. \tag{13}$$

A polynomial function on G can be thought of as a polynomial in the matrix entries: $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$, two polynomials give the same function if their difference is divisible by $ad - bc - 1$.

Let U denote the subgroup of all elements of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. We claim that any $f \in \mathbb{F}[G]$ satisfying

$$f(ug) = f(g), \forall u \in U, g \in G \tag{14}$$

is uniquely written as a polynomial in the matrix entries c and d (an easy exercise in matrix multiplication shows that a polynomial in c, d indeed defines a function satisfying (14)).

Consider the principal open subset G_d (defined by $d \neq 0$). Since G is irreducible as a variety, the restriction map $\mathbb{F}[G] \to \mathbb{F}[G_d] (= \mathbb{F}[G][d^{-1}])$ is an inclusion. Also note that G_d is preserved by the left action of U. We leave it as an exercise to show that the multiplication map

$$U \times \{\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\} \to G_d$$

is an isomorphism.

Thanks to this isomorphism, we can identify $\mathbb{F}[G_d]$ with $\mathbb{F}[z, c, d^{\pm 1}]$ so that (14) becomes the condition of being independent of z. It follows that the algebra of invariants $\mathbb{F}[G_d]^U$ is $\mathbb{F}[c, d^{\pm 1}]$. It is left as an exercise to show that the intersection of $\mathbb{F}[c, d^{\pm 1}]$ and $\mathbb{F}[G]$ in $\mathbb{F}[G_d]$ is $\mathbb{F}[c, d]$. This finishes the proof of the claim above.

Now it is easy to observe that

- The subspace in $\mathbb{F}[c, d] = \mathbb{F}[G]^U$ of functions satisfying (13) is exactly the subspace of homogeneous polynomials in c, d of degree λ.
- And the action of G (via $[gf](g') = f(gg')$) identifies this space with $M(\lambda)$, finishing the proof.

□

4.4. **Completion of Classification.**

Proof of Proposition 4.11. Our key claim is the following: Let M be a rational representation of G such that $M_\mu = \{0\}$ for all $\mu < -\lambda$. We claim that

$$M^*_{-\lambda} \cong \mathrm{Hom}_G(M, M(\lambda)), \tag{15}$$

a vector space isomorphism.

Let us first explain how (15) implies the proposition. Let V^1, V^2 be two non-isomorphic irreducible representations with highest weight λ, equivalently, lowest weight $-\lambda$. They both admit nonzero homomorphisms to $M(\lambda)$ by (15). It follows that $V^1 \oplus V^2 \hookrightarrow M(\lambda)$, hence $V^1_\lambda \oplus V^2_\lambda \hookrightarrow M(\lambda)_\lambda$. The source space is at least two-dimensional, while the target space is one-dimensional.

Now we prove (15). By the Frobenius reciprocity, Lemma 4.13 combined with Proposition 4.14 we get

$$\mathrm{Hom}_G(M, M(\lambda)) \xrightarrow{\sim} \mathrm{Hom}_B(M, \mathbb{F}_{-\lambda}). \tag{16}$$

We notice that

$$\mathrm{Hom}_B(M, \mathbb{F}_{-\lambda}) \hookrightarrow \mathrm{Hom}_T(M, \mathbb{F}_{-\lambda}) \tag{17}$$

The target is identified with $M^*_{-\lambda}$. It remains to show that (17) is an isomorphism.

Let $M_+ := \bigoplus_{\mu > -\lambda} M_\mu$. To prove the claim in the end of the previous paragraph we will check that M_+ is B-stable. It is clear that M_+ is T-stable. Since $B = T \ltimes U$, it remains to show that M_+ is U-stable. For this, we need to show that for any $m \in M_\mu$ for $\mu > -\lambda$ and any u we have $um \in M_+$. The element u is of the form as $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, we write $u(z)$ to indicate the dependence on z. We can write $u(z)m$ as $\sum_{\nu \in \mathbb{Z}} m_\nu(z)$, where m_ν is a polynomial map $\mathbb{A}^1 \to M_\nu$. Our task is to show that $m_{-\lambda} = 0$ as a polynomial map.

We have the following for all $t \in \mathbb{F}^\times, z \in \mathbb{F}$:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t^2 z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \tag{18}$$

Apply both sides of (18) to m. For the right hand side we have $t^\mu \sum_\nu m_\nu(t^2 z)$. For the left hand side we get $\sum_\nu t^\nu m_\nu(z)$. We conclude that $t^\mu m_{-\lambda}(t^2 z) = t^{-\lambda} m_{-\lambda}(z)$, equivalently,

$$\mu_{-\lambda}(t^2 z) = t^{-\mu-\lambda}\mu_{-\lambda}(z). \tag{19}$$

Since $\mu > -\lambda$, the power of t in the right hand side of (19) is negative. An easy exercise is to show that the only polynomial map satisfying (19) is zero. This finishes the proof. □

Remark 4.15. The representation $M(\lambda)$ is usually referred to as the *dual Weyl module* with highest weight λ. It dual, $W(\lambda)$, is the so called *Weyl module*. Dualizing (16) we see that

$$\mathrm{Hom}_G(W(\lambda), M) \xrightarrow{\sim} \mathrm{Hom}_B(\mathbb{F}_\lambda, M).$$

This is an analog of (1) of Proposition 2.13 for Weyl modules, so the Weyl modules may be viewed as analogs of Verma modules for rational G-representations.

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