Quantized symplectic singularities & applications to Lie theory

1) Motivation: Orbit method
2) Filtered quantizations
3) Nilpotent orbits in semisimple Lie algebras.

1) Orbit method.

Let G be a connected Lie group. A classical question in Lie Representation theory (going back to Gelfand and Harish-Chandra) is to classify irreducible unitary G-representations ("unitary" means that G acts by unitary operators in a Hilbert space w. some continuity conditions; "irreducible" means no proper closed subrepresentations - in this course unitary representations are only for motivation purposes).

There's an especially elegant answer to the classification question in the case when G is a nilpotent group, due to Kirillov (61), known as the Orbit method. Namely, for general G, we can consider the adjoint (in $\mathfrak{g} = Lie(G)$) and coadjoint (in $\mathfrak{g}^*$) G-representations

Theorem (Kirillov) Let G be nilpotent and simply connected. Then the unitary G-irreps (up to iso) are in a natural bijection with the set of G-orbits in $\mathfrak{g}^*$ (a.k.a. coadjoint orbits).

Why should one expect the coadjoint orbits to appear? This has to do with "quantization": a meta-principle that relates classical & quantum mechanical systems. In both settings one can talk about "phase spaces" (= "homes" to mechanical systems) and their symmetry. In Quantum Mechanics, a phase space is a Hilbert space, the symmetry is given by a unitary representation. The "most symmetric" situation is when the representation is irreducible.

Let's recall what these things are in Classical Mechanics. A phase space is a manifold (or a more general space) M together with a Poisson structure : an $\mathbb{R}$-bilinear map $\{\cdot,\cdot\}: C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ that satisfies the following two properties:

- it's a Lie bracket
- Leibniz identity : $\{f,gh\}=\{f,g\}h+\{f,h\}g$

For example, if ω is a symplectic (= closed & non-degenerate) 2-form, then it defines the Poisson bracket: $\{f,g\}:=\langle\omega^{-1}, df\wedge dg\rangle$, where we write ω^{-1} for the bivector field on M induced by ω. The resulting Poisson structures are called nondegenerate.

A symmetry is an action of a Lie group G on M that preserves $\{\cdot,\cdot\}$ and is "Hamiltonian" in the following sense. Note that $G\curvearrowright M \rightsquigarrow$ G-equivariant Lie algebra homomorphism $\mathfrak{g}\to \mathrm{Vect}(M)$, $\xi\mapsto \xi_M$. Namely, by a comoment map we mean a G-equivariant linear map $\varphi:\mathfrak{g}\longrightarrow C^\infty(M)$ s.t. $\xi_M=\{\varphi(\xi),\bullet\}$. We can encode it as the moment map $\mu: M \to \mathfrak{g}^*$, $<\mu(m), \xi> := [\varphi(\xi)](m)$. A Hamiltonian action of G on M is an action by Poisson automorphisms together with a (co)moment map.

Exercise 1: Show that φ is a Lie algebra homomorphism.

Example: $\mathfrak{g}^*$ is a Poisson manifold w. unique bracket satisfying $\{\xi, \eta\} = [\xi, \eta]$ $\forall \xi, \eta \in \mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$. The inclusion map $\varphi: \mathfrak{g} \hookrightarrow C^\infty(\mathfrak{g}^*)$ is a comoment map, equivalently, $id: \mathfrak{g}^* \to \mathfrak{g}^*$ is a moment map.

In this setting "most symmetric" means that $G \curvearrowright M$ is transitive. We now proceed to classifying transitive Hamiltonian actions.

Exercise 2: Here we want to construct a symplectic form on a coadjoint orbit $G\alpha \subset \mathfrak{g}^*$ and show that $G \curvearrowright G\alpha$ is Hamiltonian. Note that $\{\cdot, \cdot\}$ on $C^\infty(M)$ can be viewed as bivector $\mathcal{P} \in \Gamma(\Lambda^2 T_M)$.

1) Let $\alpha \in \mathfrak{g}^*$. Show that $\mathcal{P}_\alpha \in \Lambda^2 T_\alpha G\alpha$ and is non-degenerate there. Show that this equips $G\alpha$ w. a G-invariant Poisson structure

2) Show that the inclusion $G\alpha \hookrightarrow \mathfrak{g}^*$ is a moment map.

3) Show that the symplectic form $\omega := \mathcal{P}^{-1}$ on $G\alpha$ satisfies: $\omega_\alpha(\xi.\alpha, \eta.\alpha) = <\alpha, [\xi, \eta]>$. This is the Kirillov-Kostant form.

It turns out that a general transitive Hamiltonian action is quite close to the previous exercise:

Exercise 3: Let M be a Poisson manifold with a transitive Hamiltonian G-action. Prove that

1) $\operatorname{im}\mu \subset \mathfrak{g}^*$ is a single orbit.

2) $\mu: M \longrightarrow \operatorname{im}\mu$ is a cover & $\mu^*: C^\infty(\operatorname{im}\mu) \longrightarrow C^\infty(M)$ intertwines Poisson brackets.

3) The Poisson structure on M is nondegenerate & μ is a symplectomorphism.

Conclusion: Essentially, Orbit method predicts a "relation" between most symmetric classical & quantum mechanical systems (w. G being a group of symmetries). For G nilpotent and simply connected, the relation is a bijection. The bijection fails for G semisimple. For example, if G is compact, the unitary irreps are exactly finite dimensional ones, while the (co)adjoint orbits (for semisimple G, have $\mathfrak{g} \xrightarrow[G]{\sim} \mathfrak{g}^*$ so there's no difference between adjoint & coadjoint) are classified by points in a Weyl chamber).

While we don't study unitary representations in this course, a lot of what we do is inspired by the desire to find right analogs of Orbit method for semisimple Lie groups. For example, we'll see that if we think about quantizations algebraically, we still have an "orbit-method-like" bijection between covers of adjoint orbits (of a semisimple algebraic group$/\mathbb{C}$) and quantizations.

2) Filtered quantizations

Before we discuss an algebraic notion of a quantization let's say a couple of words about the subject of geometric representation theory. Tautologically, it seeks to study representations geometrically. This means several different (but related) things, in particular, the study of representations of algebras that have "geometric origins." Many of these algebras arise as filtered quantizations of graded Poisson algebras. This is our "algebraic version of quantization."

Setting: let A be a finitely generated commutative algebra$/\mathbb{C}$ equipped w. the following two structures:

- Poisson bracket $\{\cdot,\cdot\}: A\times A \to A$
- Vect. space grading, $A = \bigoplus_{i=0}^{\infty} A_i$, which is an algebra grading: $A_iA_j \subset A_{i+j}, \forall i,j.$

that are compatible as follows:

$$\exists\ d>0 \text{ s.t. } \deg\{\cdot,\cdot\} = -d \text{ i.e. } \{A_i, A_j\} \subset A_{i+j-d},\ \forall\, i,j.$$

Examples: 1) Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Take $A = S(\mathfrak{g})\ (= \mathbb{C}[\mathfrak{g}^*])$ w. its standard grading and the unique Poisson bracket s.t. $\{\xi,\eta\} = [\xi,\eta]\ \forall\, \xi,\eta\in\mathfrak{g}$. We have $d=1$.
2) Let V be a finite dimensional symplectic vector space w. form ω. Take $A = S(V)$ w. its standard grading and the unique Poisson bracket $\{\cdot,\cdot\}$ s.t. $\{u,v\} := \omega(u,v)\ \forall\, u,v\in V$. Here we have $d=2$.

Now let A be as in the setting above.

Definition: A (filtered) quantization of A is a pair $(\mathcal{A},\iota)$, where

• $\mathcal{A}$ is an associative algebra w. filtration $\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_{\leq i}$ s.t. $\mathcal{A}_{\leq i} \mathcal{A}_{\leq j} \subset \mathcal{A}_{\leq i+j}$ and $[\mathcal{A}_{\leq i}, \mathcal{A}_{\leq j}] \subset \mathcal{A}_{\leq i+j-d}$ (w. $[a,b] := ab - ba$). It follows that the algebra $\operatorname{gr}\mathcal{A} = \bigoplus_i \mathcal{A}_{\leq i}/\mathcal{A}_{\leq i-1}$ is commutative & has Poisson bracket given by taking the top degree of $[\cdot,\cdot]$, i.e. $\{a + \mathcal{A}_{\leq i-1}, b + \mathcal{A}_{\leq j-1}\} = [a,b] + \mathcal{A}_{\leq i+j-d-1}$.

Exercise 1 – check this is indeed a Poisson bracket.

• $\iota: \operatorname{gr}\mathcal{A} \xrightarrow{\sim} A$ is a graded Poisson algebra isomorphism.

Definition: By an isomorphism of quantizations $(\mathcal{A}, \iota)$, $(\mathcal{A}', \iota')$ of A we mean a filtered algebra isomorphism $\varphi: \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ s.t. $\iota' \circ \operatorname{gr}\varphi = \iota$, where $\operatorname{gr}\varphi: \operatorname{gr}\mathcal{A} \xrightarrow{\sim} \operatorname{gr}\mathcal{A}'$ is an isomorphism induced by φ.

Example: 1) $U(\mathfrak{g})$ is a quantization of $S(\mathfrak{g})$ – PBW theorem.

2) The Weyl algebra $W(V) = T(V)/(u \otimes v - v \otimes u - \omega(u,v) \mid u, v \in V)$ is a (unique, in fact) quantization of $S(V)$ (exercise: check this).

Problem: for given A classify its (filtered) quantizations up to iso.

This problem doesn't have a good answer unless we impose some restrictions on A. Our restriction has to do with "symplectic singularities." Luckily, that's exactly the class of Poisson algebras that we should actually care about in this course. This notion will be defined in the next lecture. An important family of varieties with symplectic singularities arises from nilpotent orbits in semisimple Lie algebras, to be discussed next.

3) Nilpotent orbits in semisimple Lie algebras.

Let G be a semisimple algebraic group$/\mathbb{C}$ & $\mathfrak{g}=\mathrm{Lie}(G)$. Consider the adjoint action $G \curvearrowright \mathfrak{g}$. We are interested in understanding its orbits. Note that $\mathfrak{g} \simeq_G \mathfrak{g}^*$. So we are talking about coadjoint orbits, the classical side of the Orbit method, see Sec 1. In particular, all adjoint orbits are symplectic, so, even dimensional.

Definition: $\xi \in \mathfrak{g}$ is called *nilpotent* if it's represented by a nilpotent operator in a faithful finite dimensional representation.

Fact: This is independent of the choice of a faithful representation.

Remark: 1) $\mathfrak{g}=\mathfrak{sl}_n, \mathfrak{so}_n$ or $\mathfrak{sp}_n$: ξ is nilpotent $\Leftrightarrow$ ξ is a nilpotent matrix.

2) ξ is nilpotent $\Leftrightarrow$ $\mathrm{Ad}(g)\xi$ is $\forall g \in G$. So we can talk about *nilpotent orbits* in $\mathfrak{g}$.

We will be interested in the classification of nilpotent orbits. A key tool is to relate them to homomorphisms $\mathfrak{sl}_2 \to \mathfrak{g}$.

Definition: An *$\mathfrak{sl}_2$-triple* in $\mathfrak{g}$ is a triple $e,h,f \in \mathfrak{g}$ satisfying the defining relations of $\mathfrak{sl}_2$, i.e. $\exists$ homomorphism $\varphi: \mathfrak{sl}_2 \to \mathfrak{g}$ w.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f.$$

Exercise: e ($\&$ f) is nilpotent.

A connection between nilpotent orbits & $\mathfrak{sl}_2$-triples is as follows.

Theorem 1 (Jacobson-Morozov): $\forall$ nilpotent $e \in \mathfrak{g}$ $\exists$ $\mathfrak{sl}_2$-triple containing e.

Theorem 2 (Kostant): If (e,h,f), (e,h',f') are $\mathfrak{sl}_2$-triples then $\exists$ $g \in G$ s.t. $Ad(g)(e,h,f) = (e,h',f')$.

Corollary: There's a bijection between:

- nilpotent G-orbits
- G-conjugacy classes of $\mathfrak{sl}_2$-triples (e,h,f)

given by $(e,h,f) \mapsto e$.

Example: $G = SL_n(\mathbb{C})$. The G-conjugacy classes of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ is the same thing as isomorphism classes of n-dimensional $\mathfrak{sl}_2$-reps. By the classification of finite dimensional $\mathfrak{sl}_2$-reps, the n dimensional one are classified by the partitions of n: to $V = \bigoplus_{i=1}^{k} V(d_i)$ (where $V(d_i)$ is the d_i-dimensional irrep) we assign the partition w. parts d_i. Since e acts on an irrep by a single Jordan block, we recover the classification of conjugacy classes of nilpotent matrices by Jordan types.

There's a similar although more complicated story for other classical Lie algebras. Let $G = Sp_n(\mathbb{C})$ (n even) or $O_n(\mathbb{C})$ – which is disconnected.

Proposition: The nilpotent G-orbits in $\mathfrak{g}$ are classified by the partitions of n where each odd (for Sp)/even (for O) part occurs w. even multiplicity (via Jordan type).

A proof is sketched in the Exercise sheet.

Remark: Since $SO_n(\mathbb{C})$ has index 2 in $O_n(\mathbb{C})$, an $O_n(\mathbb{C})$-orbit may split into the disjoint union of two $O_n(\mathbb{C})$ or remain a single orbit. The former happens iff all parts are even (exercise).

In Sec 2 we've talked about filtered quantizations of finitely generated graded Poisson algebras. We now produce a family of examples of such algebras coming from nilpotent orbits in s/simple Lie algebras.

Theorem: Let $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit. Then $\mathbb{C}[\mathbb{O}]$ is a finitely generated graded Poisson algebra (w. $\deg\{\cdot,\cdot\} = -1$).

Proof: • The Poisson structure arises b/c $\mathbb{O}$ is symplectic (see the 1st paragraph of Sec 3 in Lec 1).

• To establish the grading & prove $\mathbb{C}[\mathbb{O}]$ is fin. gen. we need a consequence of the theory of $\mathfrak{sl}_2$-triples that we've already seen for $\mathfrak{g}$ classical.

Fact 1: # of nilpotent orbits in $\mathfrak{g}$ is finite.

• $\mathbb{C}[\mathbb{O}]$ is finitely generated.

Exercise (in Exer sheet 2)
The nilpotent cone $\mathcal{N} := \{\text{nilpotent } \xi \in \mathfrak{g}\}$ is Zariski closed in $\mathfrak{g}$.

So, $\overline{\mathbb{O}} \setminus \mathbb{O}$ consists of nilpotent orbits. Their number is finite & all of them are even dimensional. So $\operatorname{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}} \setminus \mathbb{O} \geqslant 2$. And $\mathbb{O}$ is smooth. Now we are done by the following fact from Alg. geometry/Comm. algebra:

Fact 2: Let X be an irreducible affine variety, $X^\circ \subset X$ a smooth open subvariety w. $\operatorname{codim}_X X \setminus X^\circ \geqslant 2$. Then $\mathbb{C}[X^\circ]$ is the normalization (=integral closure in the fraction field) of $\mathbb{C}[X]$, hence is finitely generated.

• $\mathbb{C}[\mathbb{O}]$ is graded: consider the dilation action $\mathbb{C}^\times \curvearrowright \mathfrak{g}$: $t.\xi = t^{-1}\xi$. Note that $\mathcal{N}$ is $\mathbb{C}^\times$-stable.
Exercise : $\forall$ orbit $\mathbb{O} \subset \mathcal{N}$ is $\mathbb{C}^\times$-stable (hint: use that there are finitely many nilpotent orbits; or $SL_2 \to G$ coming from $\mathfrak{sl}_2$-triple).

$\mathbb{C}^\times \curvearrowright \mathbb{O} \rightsquigarrow$ grading on $\mathbb{C}[\mathbb{O}]$. The following exercise finishes the proof.
Exercise: Use 3) of Exercise 2 in Sec 1 of Lec 1 to show that $\deg \{\cdot,\cdot\} = -1$. $\square$