Quantized symplectic singularities & applications to Lie theory, Lecture 2.

1) Equivariant covers of nilpotent orbits.
2) Singular symplectic varieties.
3) Classification of filtered quantizations.

1) Let $\mathbb{O}\subset\mathfrak{g}$ be a nilpotent orbit. Then $\mathbb{O}\simeq G/H$ w. $H=Z_G(e)$. A G-equivariant cover of $\mathbb{O}$ has the form G/H' w. $H'\subset H$, a finite index subgroup. In other words, covers are parameterized by subgroups of H/H°, where H° is the connected component of 1 in H. In what follows we will often call $\widetilde{\mathbb{O}}:=G/H'$ a nilpotent cover. We'd like to understand the group H/H°. This is done using $\mathfrak{sl}_2$-triples.

Exercise: • Show that $Z_G(e)\simeq Z_G(e,h,f)\ltimes$ unipotent group.

• Deduce $Z_G(e)/Z_G(e)^\circ \xrightarrow{\sim} Z_G(e,h,f)/Z_G(e,h,f)^\circ$.

The component group $Z_G(e)/Z_G(e)^\circ$ is known in all cases. For classical Lie algebras it's easy to determine $Z_G(e,h,f)$. This is done in the exercise sheet for Lecture 1 for BCD types

Proposition: Let $G=SL_n(\mathbb{C}), O_n(\mathbb{C})$ or $Sp_n(\mathbb{C})$. Let $\mathbb{O}$ be a nilpotent orbit corresponding to a partition $(n_1^{d_1},\dots,n_k^{d_k})$ ($d_i>0$ is the multiplicity) & $e\in\mathbb{O}$.

1) Let $G=SL_n(\mathbb{C})$. Then $Z_G(e,h,f)\simeq\{(g_1,\dots,g_k)\in\prod GL(d_i)\mid \prod\det(g_i)^{n_i}=1\}$ and $Z(G)(\simeq\mathbb{Z}/n\mathbb{Z})\twoheadrightarrow Z_G(e)/Z_G(e)^\circ\simeq\mathbb{Z}/GCD(n_1,\dots,n_k)\mathbb{Z}$.

2) Let $G = O_n(\mathbb{C})$ or $Sp_n(\mathbb{C})$. Then $Z_G(e,h,f) \simeq \prod_{i=1}^{k} G_i$, where $G_i \simeq O_{d_i}$ if n_i is odd & Sp_{d_i} if n_i is even (for O_n; vice versa for Sp_n). Therefore, $Z_G(e)/Z_G(e)^\circ \simeq (\mathbb{Z}/2\mathbb{Z})^a$, where $a =$ # of odd (for O_n)/ even (for Sp_n) n_i's (= # of O factors in $Z_G(e,h,f)$)

Example: $G = Sp_{2n}(\mathbb{C})$, $\mathbb{O}$ corresponds to $(2,1,\dots,1)$. We have $Z_G(e)/Z_G(e)^\circ \simeq \mathbb{Z}/2\mathbb{Z}$. We claim that the 2-fold cover $G/Z_G(e)^\circ$ is $\mathbb{C}^{2n}\setminus\{0\}$. Namely, consider the natural G-action on $\mathbb{C}^{2n}$. This action is Hamiltonian w. moment map $\mu: \mathbb{C}^{2n} \to \mathfrak{g}^*$: $\langle \mu(v), \xi\rangle = \frac{1}{2}\omega(\xi v, v)$, where ω is the form used to define G.

Exercise: 1) Check this.

2) Show that $\operatorname{im}\mu = \overline{\mathbb{O}}$

3) Show that over $\mathbb{O}$ the morphism μ is a 2-fold cover.

So, we get the conclusion of this example.

In the end of Sec 3 of Lec 1 we have shown that the algebra $\mathbb{C}[\mathbb{O}]$ is finitely generated for all nilpotent orbits $\mathbb{O}$. This generalizes to all nilpotent covers.

Theorem: Let $\widetilde{\mathbb{O}}$ be an equivariant cover of a nilpotent orbit $\mathbb{O}$. Then $\mathbb{C}[\widetilde{\mathbb{O}}]$ is a finitely generated graded Poisson algebra.

Sketch of proof: • $\widetilde{\mathbb{O}}$ is symplectic: discussed in Sec 1 of Lec 1.

• $\mathbb{C}[\widetilde{\mathbb{O}}]$ is fin. gen'd: consider the "Stein decomposition" for $\widetilde{\mathbb{O}} \longrightarrow \overline{\mathbb{O}}$:

it factorizes as the composition $\widetilde{\mathbb{O}} \to X \twoheadrightarrow \overline{\mathbb{O}}$, where $X =$ Spec of the integral closure of $\mathbb{C}[\overline{\mathbb{O}}]$ in the field of rational functions on $\widetilde{\mathbb{O}}$, and $\widetilde{\mathbb{O}} \to X$ is an open embedding. From $\operatorname{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}} \setminus \mathbb{O} \geqslant 2$, we deduce $\operatorname{codim}_X X \setminus \widetilde{\mathbb{O}} \geqslant 2$ and by Fact 2 in Sec 3 of Lec 1, we get $\mathbb{C}[\widetilde{\mathbb{O}}] = \mathbb{C}[X]$.

• Grading $\rightsquigarrow \mathbb{C}^\times \curvearrowright \widetilde{\mathbb{O}}$ lifted from $\mathbb{C}^\times \curvearrowright \mathbb{O}$ by $z.\xi = z^d \xi$ for suitable $d > 0$. $\square$

Def'n: the affinization of $\widetilde{\mathbb{O}}$ is $X := \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$

2) Singular symplectic varieties.

2.1) Definition:

We can talk about symplectic smooth varieties: these are smooth algebraic varieties equipped with an algebraic symplectic form: a symplectic vector space is the most basic example.

Every symplectic smooth variety X is Poisson meaning that $\mathcal{O}_X$ comes w. a Poisson bracket.

Beauville (2000) generalized the notion of "symplectic" to singular Poisson varieties.

Definition: Let X be a Poisson variety. We say X is symplectic (a.k.a. singular symplectic, a.k.a. has symplectic singularities) if

i) X is normal (and, for simplicity of exposition, irreducible),

ii) the restriction of the Poisson structure to the smooth locus $X^{reg} \subset X$ is non-degenerate. Let $\omega^{reg} \in \Omega^2(X^{reg})$ be the corresponding symplectic form, &

iii) there's a resolution of singularities $\pi: Y \to X$ (meaning that Y is smooth and π is birational & proper) s.t. $\pi^*(\omega^{reg})$ extends from $\pi^{-1}(X^{reg})$ to a regular 2-form on Y.

Remarks: 1) if we have iii) for one resolution, it's true for any resolution, as proved already by Beauville.

2) The extension of $\pi^*(\omega^{reg})$ to Y is closed but may fail to be non-degenerate. If it's nondegenerate ($\Leftrightarrow$ symplectic) we say that $\pi: Y \to X$ is a symplectic resolution.

2.2) Examples: symplectic quotient singularities.

Let V be a finite dimensional symplectic vector space w. form ω and $\Gamma \subset Sp(V)$ be a finite subgroup. Set $X = V/\Gamma$ $(= \operatorname{Spec} S(V)^\Gamma)$. It's a Poisson variety b/c $S(V)^\Gamma \subset S(V)$ is closed under $\{\cdot,\cdot\}$. Beauville proved that it's singular symplectic. Namely, the claim that X is normal is standard. If $\eta: V \to V/\Gamma$ is the quotient morphism, then $(V/\Gamma)^{reg}$ is symplectic: the symplectic form descends from the restriction of ω to $\eta^{-1}((V/\Gamma)^{reg}) \subset V$ b/c η is unramified over $(V/\Gamma)^{reg}$.

The claim that V/Γ satisfies (iii) was checked by Beauville.

Sometimes V/Γ has a symplectic resolution (and it's mostly known when). Notable examples:

• $\dim V = 2$ so that $\Gamma \subset SL_2(\mathbb{C})$. The symplectic resolution of $\mathbb{C}^2/\Gamma$ is the unique minimal resolution. This case is very important for the general theory b/c "locally" every singular symplectic variety of dim 2 is $\mathbb{C}^2/\Gamma$ for suitable Γ.

• This case isn't relevant for this course but too important to ignore: $V = (\mathbb{C}^2)^{\oplus n}$, $\Gamma = S_n$ acting on V by permuting the copies of $\mathbb{C}^2$. Then X is the symmetric power $(\mathbb{C}^2)^{\oplus n}/S_n$ (parameterizes unordered n-tuples of points in $\mathbb{C}^2$), while for Y we can take $\mathrm{Hilb}_n(\mathbb{C}^2)$ (parameterizing length n subschemes of $\mathbb{C}^2$). It's symplectic.

2.3) Examples: Spec $\mathbb{C}[\widetilde{\mathbb{O}}]$

Let G be a semisimple algebraic group, $\widetilde{\mathbb{O}}$ be a nilpotent cover.

Thm: $\widetilde{X} := \mathrm{Spec}\ \mathbb{C}[\widetilde{\mathbb{O}}]$ is (singular) symplectic.

The case of $\widetilde{\mathbb{O}} \hookrightarrow \mathfrak{g}$ follows from the work of Panyushev & Hinitch: one can produce Y using the theory of $\mathfrak{sl}_2$-triples, this is done in Exercise sheet. The general case can be deduced from here using some results from Algebraic geometry.

Sometimes $\widetilde{X}$ admits a symplectic resolution. Turns out, most of such resolutions are of the form $T^*(G/P)$, where $P \subset G$ is parabolic. Namely, a parabolic subgroup of G is an (automatically connected) algebraic subgroup P containing a Borel subgroup $\Leftrightarrow G/P$ is projective. We have a decomposition $P = L \ltimes U$, where L is connected reductive (Levi subgroup) & U is unipotent. For example, for $G = SL_n$, and a composition $n = n_1 + \dots + n_k$ we can consider the subgroup P of block upper triangular matrices w. block sizes $n_1, \dots n_k$. It's parabolic w. Levi L of all block-diagonal matrices & U being the kernel of the natural projection $P \twoheadrightarrow L$. Let $\mathfrak{n} := \mathrm{Lie}(U)$.

Consider the cotangent bundle $T^*(G/P)$. It's smooth & symplectic. It can be thought of as the homogeneous vector bundle $G \times^P (\mathfrak{g}/\mathfrak{p})^*$ (the quotient of $G \times (\mathfrak{g}/\mathfrak{p})^*$ by the P-action given by $p.(g,\alpha) = (gp^{-1}, Ad^*(p)\alpha)$). We write $[g,\alpha]$ for the P-orbit of (g,α), this is a point in $G \times^P (\mathfrak{g}/\mathfrak{p})^*$. Note that $\mathfrak{p}^\perp = \mathfrak{n}$ w.r.t. Killing form giving a P-equivariant isomorphism $(\mathfrak{g}/\mathfrak{p})^* \xrightarrow[P]{\sim} \mathfrak{n}$. The G-action on $T^*(G/P)$ is Hamiltonian w. moment map given by $\mu([g,\alpha]) = Ad(g)\alpha$.

Exercise: μ is proper.

Every element in $\mathfrak{n}$ is nilpotent, so $\mathrm{im}\,\mu$ consists of nilpotent elements. Since the number of nilpotent orbits is finite (Fact 1 in Sec 1 of Lec 3) & $\mathrm{im}\,\mu$ is irreducible (b/c $T^*(G/P)$ is), it follows that $\exists!$ orbit $\mathbb{O}_P \subset \mathcal{N}$ s.t. $\mathrm{im}\,\mu = \overline{\mathbb{O}}_P$. In the next lecture, we'll sketch the argument that $\dim \mathbb{O}_P = \dim T^*(G/P)$. It follows that $T^*(G/P)$ contains a unique open orbit, $\widetilde{\mathbb{O}}_P$, which is a G-equivariant cover of $\mathbb{O}_P$. By the Stein factorization we have the following commutative diagram:

$$\begin{tikzcd} T^*(G/P) \arrow[r] \arrow[d, "\pi"'] & \mathfrak{g} \\ \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}] \arrow[r, two heads] & \overline{\mathbb{O}} \arrow[u, hook] \end{tikzcd}$$

Exercise: π is a symplectic resolution of singularities.

Examples: 1) Let $P=B$. Then $\operatorname{im}\mu = \mathcal{N}$. According to Kostant, $\mathcal{N}$ is normal. So we get a symplectic resolution $T^*(G/B) \to \mathcal{N}$, the Springer resolution. This is one of the most important morphisms in the geometric Representation theory – see the Chriss-Ginzburg book.

2) Let $\mathfrak{g} = \mathfrak{sl}_n$. Take the nilpotent orbit $\mathbb{O} = \mathbb{O}_\lambda$ for $\lambda \vdash n$. Let λ^t denote the transposed partition (i.e. the partition corresponding to the transposed Young diagram). Pick the composition consisting of parts of λ in some order and take the corresponding parabolic, P. One can show that $\operatorname{im}\mu = \overline{\mathbb{O}}$ & μ is generically injective. Moreover, Kraft and Procesi checked that $\overline{\mathbb{O}}$ is normal. It follows that $\pi: T^*(G/P) \twoheadrightarrow \overline{\mathbb{O}}$ is a symplectic resolution.

3) Let $G = Sp_4$ and $\mathbb{O}$ correspond to the partition $(2,2)$. The group $Z_G(e)/Z_G(e)^\circ$ is $\mathbb{Z}/2\mathbb{Z}$ by Proposition in Sec 1.2. So we have a 2-fold cover $\tilde{\mathbb{O}}$ of $\mathbb{O}$. On the other hand, we have 2 semisimple rk 1 parabolics, P_1, P_2, corresponding to long & short roots.

Exercise: Show that $T^*(G/P_1)$ and $T^*(G/P_2)$ are symplectic resolutions of Spec $\mathbb{C}[\mathbb{O}]$ & Spec $\mathbb{C}[\tilde{\mathbb{O}}]$ – and figure out what resolves what.

3) Classification of filtered quantizations.

Setting: A: finitely generated graded commutative $\mathbb{C}$-algebra w. Poisson bracket of degree $-d$ ($d \in \mathbb{Z}_{>0}$). We assume that

- $A_0 = \mathbb{C}$
- $X := \operatorname{Spec}(A)$ is singular symplectic

$\Big\}$ say: X is a conical symplectic singularity.

Theorem (I.L. 2016): There are a finite dimensional $\mathbb{C}$-vector space $\mathfrak{h}_X$ & a finite crystallographic reflection group $W_X \subset GL(\mathfrak{h}_X)$ s.t. there's a natural identification:

$$\{\text{quantizations of } A\}/\text{iso} \xrightarrow{\sim} \mathfrak{h}_X/W_X.$$

Example: Let $\mathfrak{g}$ be a s/simple Lie algebra and $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone, by Sec 2.3, $\mathcal{N}$ is singular symplectic. Let $\mathfrak{h}, W$ be a Cartan subalgebra & Weyl group of $\mathfrak{g}$. Then $\mathfrak{h}_{\mathcal{N}} = \mathfrak{h}^*$, $W_{\mathcal{N}} = W$. The quantization corresponding to $W\lambda \in \mathfrak{h}^*/W$ is constructed as follows.

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ & $\mathcal{Z} \subset \mathcal{U}(\mathfrak{g})$ be its center. We have the Harish-Chandra isomorphism $\mathcal{Z} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$ (where, by convention, $W \curvearrowright \mathfrak{h}^*$ is the <u>usual</u> linear action). So to $\lambda \in \mathfrak{h}^*$ we can assign the maximal ideal $m_\lambda \in \mathbb{C}[\mathfrak{h}^*]^W \simeq \mathcal{Z}$ consisting of all functions vanishing at λ, of course, $m_\lambda = m_{w\lambda}$ $\forall\, w \in W$.

Then the quantization of $\mathbb{C}[\mathcal{N}]$ corresponding to $W\lambda$ is the "central reduction" $\mathcal{U}_\lambda := \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{m}_\lambda$. A proof why it's indeed a quantization is sketched in Exercise sheet.

Then we have the following two questions:

1) How to construct $\mathfrak{h}_X$?

2) For $\lambda \in \mathfrak{h}_X$, how to construct the corresponding quantization?

We'll start by answering 1) in a special case, the general case will be handled in Lec 3. Question 2) will be addressed in Lec 4.

The simplest case of question 1) is when X admits a symplectic resolution, say Y. Then

$$\mathfrak{h}_X := H^2(Y, \mathbb{C}).$$

Example: Let $X = \mathcal{N}$, we can take $Y = T^*(G/B)$. We have $H^2(T^*(G/B), \mathbb{C}) = H^2(G/B, \mathbb{C}) =$ [for simply connected Lie group G & Lie subgroup F, have $H^2(G/F, \mathbb{C}) = H^1(F^\circ, \mathbb{C})^{F/F^\circ}$] $= H^1(B, \mathbb{C}) = \mathfrak{h}^*$.

We'll see that in the general case we need to modify Y.