Quantized symplectic singularities & applications to Lie theory, Lecture 3.

1) $\mathbb{Q}$-factorial terminalizations.
2) The case of nilpotent covers.
3) Complements.

1) $\mathbb{Q}$-factorial terminalizations.

Below X is a conical symplectic singularity, i.e. X is (singular) symplectic & $A=\mathbb{C}[X]$ is graded so that $A=\bigoplus_{i\geqslant 0} A_i$ w. $A_0=\mathbb{C}$ & $\deg\{\cdot,\cdot\}=-d\in\mathbb{Z}_{<0}$. In Section 3 of Lec 2 we have stated that filtered quantizations of A are parameterized by pts of a certain vector space, $\mathfrak{h}_X$ (modulo a finite group action). We have stated that if Y is a symplectic resolution of X, then $\mathfrak{h}_X=H^2(Y,\mathbb{C})$. In this lecture we'll explain what happens in the general case (and in a bonus section explain how to recover $\mathfrak{h}_X$ from X).

If X doesn't have a symplectic resolution, the story is more complicated. An arbitrary resolution won't help (e.g. H^2's depend on the resolution). Instead, we should be looking at a "maximal <u>partial</u> Poisson resolution" $\pi: Y\to X$, where

- Y is a Poisson variety, possibly singular
- π is a birational proper map that is Poisson meaning that

for $f,g \in \mathbb{C}[X] \Rightarrow \{\pi^*(f), \pi^*(g)\} = \pi^*(\{f,g\})$.

• "Maximal" means that if Y' is another Poisson variety & $\pi': Y' \to Y$ another proper birational Poisson morphism, then π' is an isomorphism. For example, this implies Y is normal: otherwise for $\pi': Y' \to Y$ take the normalization morphism. One can show that $\{\cdot,\cdot\}$ extends from $\mathcal{O}_Y$ to $\mathcal{O}_{Y'}$, making π' a Poisson morphism, and it's birational & proper.

Exercise: Y is singular symplectic

Hint: consider a resolution $\tilde{\pi}: \tilde{Y} \to Y$ that is iso over Y^{reg}. $\tilde{\pi} \circ \pi: \tilde{Y} \to X$ is a resolution of singularities for X so $(\tilde{\pi} \circ \pi)^* \omega^{reg}$ extends to $\tilde{Y}$. From π being Poisson deduce that $\pi^* \omega^{reg}$ extends to a <u>symplectic</u> form on Y^{reg}.

Remark: In particular, the exercise implies that a symplectic resolution has the maximality property (if it exists).

It turns out that a maximal partial Poisson resolution always exists (nontrivial). Moreover, it admits a transparent algebro-geometric characterization: it's "$\mathbb{Q}$-factorial" & "terminal."

To define the former recall that to a scheme Z we can assign its Picard group $\mathrm{Pic}(Z)$ whose elements are isomorphism classes of line bundles on Z.

Definition: Let Z be an irreducible normal variety. We say Z is $\mathbb{Q}$-factorial if the cokernel of the restriction map $\mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(Z^{reg})$ is torsion.

Example 1: Let's investigate when $X = \mathrm{Spec}\,\mathbb{C}[\tilde{\mathbb{O}}]$ is $\mathbb{Q}$-factorial. First of all, a version of the Nakayama lemma shows that $\mathrm{Pic}(X) = \{0\}$ (here we use that X is conical).

Second, $\tilde{\mathbb{O}} \hookrightarrow X$ w. complement of codim ≥ 2. This shows that $\mathrm{Pic}(X^{reg}) \xrightarrow{\sim} \mathrm{Pic}(\tilde{\mathbb{O}})$. One can compute $\mathrm{Pic}(\tilde{\mathbb{O}})$ as follows. Assume G is simply connected. Let $\tilde{\mathbb{O}} \simeq G/H$. Then $\mathrm{Pic}(\tilde{\mathbb{O}})$ is identified w. the character group $\mathfrak{X}(H) := \mathrm{Hom}(H, \mathbb{C}^\times)$. So X is $\mathbb{Q}$-factorial iff $\mathfrak{X}(H)$ is finite. Let $\mathbb{O} \subset \mathfrak{g}$ be the orbit covered by $\tilde{\mathbb{O}}$ & $e \in \mathbb{O}$. Then $H = Q \ltimes U$, where Q is a finite index subgroup in $Z_G(e,h,f)$. Using our knowlenge of $Z_G(e,h,f)$ in Exercise sheet 1 we see that:

- if $G = SL_n$, then X is $\mathbb{Q}$-factorial $\Leftrightarrow$ all parts in the partition of $\mathbb{O}$ are equal.
- if $G = SO_{2n+1}$ or Sp_{2n}, and $\tilde{\mathbb{O}} = \mathbb{O}$, then X is $\mathbb{Q}$-factorial.

If $G = SO_{2n}$ & $\tilde{\mathbb{O}} = \mathbb{O}$, then X is $\mathbb{Q}$-factorial for most $\mathbb{O}$.

Now we explain what "terminal" means in our setting.

Definition/proposition (Namikawa): We say that a singular symplectic variety Z is terminal if $\mathrm{codim}_Z Z^{sing} \geq 4$.

Example 3: Let $\mathfrak{g}$ be a classical Lie algebra so that the nilpotent orbits are classified by (certain) partitions. Let $\mathbb{O}\subset\mathfrak{g}$ be a nilpotent orbit & $X=\operatorname{Spec}\mathbb{C}[\mathbb{O}]$. It turns out that X is terminal $\Leftrightarrow$ $\operatorname{codim}_{\overline{\mathbb{O}}}\overline{\mathbb{O}}\setminus\mathbb{O}\geqslant 4$ ($\Leftarrow$ is easy, $\Rightarrow$ not quite). One can analyze the latter condition combinatorially. The result is that X is terminal $\Leftrightarrow$ the partition $\lambda=(\lambda_1,\ldots\lambda_k)$ corresponding to $\mathbb{O}$ satisfies $\lambda_i\leqslant\lambda_{i+1}+1\ \forall\ i$ (w. $\lambda_i=0$ for $i\gg 0$, by convention). Two ingredients: the dimension formula for orbits & a description of the order by orbit closures.

We have the following result, a consequence of a more general result of Birkar-Cascini-Hacon-McKernan on MMP.

Theorem: A maximal partial Poisson resolution Y of X exists & is $\mathbb{Q}$-factorial & terminal. Conversely, any $\mathbb{Q}$-factorial & terminal partial Poisson resolution is maximal.

It turns out that many questions about X can be answered by studying Y. The computation of $\mathfrak{h}_X$ is just one of them.

Fact: We have $\mathfrak{h}_X=H^2(Y^{reg},\mathbb{C})$.

Note that this generalizes the description of $\mathfrak{h}_X$ in the case when X admits a symplectic resolution.

2) $\mathbb{Q}$-factorial terminalizations for nilpotent covers.

Here we are going to explain how to construct the $\mathbb{Q}$-factorial terminalization Y for $X := \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$. For this we need a suitable version of the so called Lusztig-Spaltenstein induction (a version of parabolic induction in this setting)

Pick a Levi subgroup $L \subset G$ and include it into a parabolic subgroup $P = L \ltimes U$. Take an L-equivariant cover $\tilde{\mathbb{O}}_L$ of a nilpotent orbit in $\mathfrak{l} (\simeq \mathfrak{l}^*)$. Set $X_L = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}_L]$. From the pair (P, X_L) we will produce an "induced" singular symplectic variety Y w. Hamiltonian G-action. For $X_L = \{0\}$ we recover $T^*(G/P) = G \times^P (\mathfrak{g}/\mathfrak{p})^*$.

Recall that $\operatorname{Lie}(U) = \mathfrak{n}$.

• Construction of Y: For this we will use the general construction known as Hamiltonian reduction.

Consider the cotangent bundle T^*G. The action $G \times G \curvearrowright G$ lifts to a Hamiltonian $G \times G$-action on T^*G. The moment map for the latter can be described as follows. We identify $T^*G \xrightarrow{\sim} G \times \mathfrak{g}^*$ using the left invariant vector fields on G, so that the G-action becomes $(g_1, g_2).(g, \alpha) = (g_1 g g_2^{-1}, g_2.\alpha)$. The moment map μ^r (resp., μ^ℓ) for the G-action on the right (resp. on the left) is $(g, \alpha) \mapsto -\alpha$ (resp. $(g, \alpha) \mapsto g.\alpha$). The action of L on X_L is also Hamiltonian, the moment map $\mu_L : X_L \to \mathfrak{l}^*$ is finite: it's the composition $X_L \twoheadrightarrow \overline{\mathbb{O}}_L \hookrightarrow \mathfrak{l}^*$. We can inflate the L-action on X_L to a P-action, and view μ_L as a moment map for the P-action by composing it w. $\mathfrak{l}^* \hookrightarrow \mathfrak{l}^* \oplus \mathfrak{n}^* = \mathfrak{p}^*$

Consider the Poisson variety $T^*G \times \tilde{X}_L$ w. diagonal action of P. It's Hamiltonian w. moment map $\mu: (g,\alpha,x) \mapsto -\alpha|_{\mathfrak{p}} + \mu_L(x)$.

Consider $\mu^{-1}(0) = \{(g,\alpha,x) \mid \alpha|_{\mathfrak{p}} = \mu_P(x)\}$. The locus $\{(\alpha,x) \mid \alpha|_{\mathfrak{p}} = \mu_L(x)\}$ is identified w. $X_L \times (\mathfrak{g}/\mathfrak{p})^*$: we send (α,x) to $(x, \alpha - \mu_L(x))$. For a suitable P-action on $X_L \times (\mathfrak{g}/\mathfrak{p})^*$ the identification

$$\mu^{-1}(0) \xrightarrow{\sim} G \times X_L \times (\mathfrak{g}/\mathfrak{p})^*$$

is P-equivariant, technical details are left as an exercise.

The action $P \curvearrowright \mu^{-1}(0)$ admits a quotient variety to be denoted by $\mu^{-1}(0)/P$. It's the homogeneous bundle over G/P w. fiber $X_L \times (\mathfrak{g}/\mathfrak{p})^*$, hence another notation: $G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*)$.

A typical point in $G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*)$ will be denoted by $[g,x,\beta]$, this is, by definition, the P-orbit of $(g,x,\beta) \in G \times X_L \times (\mathfrak{g}/\mathfrak{p})^*$.

- Poisson structure on Y & moment map for $G \curvearrowright Y$.

The bracket on $\mathcal{O}_Y$ is also produced by Hamiltonian reduction:

Important exercise: Let A be a Poisson algebra equipped w. a rational action of an algebraic group G by Poisson algebra automorphisms. Assume, further that there is a comoment map $\varphi: \mathfrak{g} \to A$, i.e. a G-equivariant linear map w. $\{\varphi(\xi), \cdot\} = \xi_A$ (the derivation of A coming from the G-action). Then $\exists$ Poisson bracket on $(A/A\varphi(\mathfrak{g}))^G$ s.t. $\{a + A\varphi(\mathfrak{g}), b + A\varphi(\mathfrak{g})\} = \{a,b\} + A\varphi(\mathfrak{g})$, $\forall\, a + A\varphi(\mathfrak{g}), b + A\varphi(\mathfrak{g}) \in (A/A\varphi(\mathfrak{g}))^G$.

The Poisson algebra $(A/A\varphi(\mathfrak{g}))^G$ is called the Hamiltonian reduction of A.

To apply this construction in our setting let $\varpi: G\to G/P$, $\eta: \mu^{-1}(0)/P = G\times^P(X_L\times(\mathfrak{g}/\mathfrak{p})^*)\to G/P$ denote the projection maps.

Exercise: For an open affine subvariety $U\subset G/P$, the algebra $\mathbb{C}[\eta^{-1}(U)]$ coincides w. the Hamiltonian reduction for $P\curvearrowright \mathbb{C}[T^*\varpi^{-1}(U)\times X_L]$.

This and the important exercise show that we have a well-defined Poisson bracket on $\mathcal{O}_Y$ (glued from $\{\cdot,\cdot\}$'s on $\mathbb{C}[\eta^{-1}(U)]$ obtained by reduction).

Now we proceed to the moment map for $G\curvearrowright Y$. Note that $\mu^{-1}(0)$ is G-stable under the left G-action on $T^*G\times X_L$ and so G acts on $\mu^{-1}(0)/P$: $g.[g', x, \beta] = [gg', x, \beta]$. There is a P-invariant & G-equiv't map $\mu': \mu^{-1}(0)\longrightarrow \mathfrak{g}^*$: $(g, x, \beta)\mapsto g.(\mu_L(x)+\beta)$. It descends to a G-equivariant map $\mu^{-1}(0)/P\longrightarrow\mathfrak{g}^*$ also to be denoted by μ'.

Exercise:
- μ' is proper.
- μ' is a moment map for $G\curvearrowright Y$.
- $\operatorname{im}\mu'$ is the closure of a single nilpotent orbit.

- $\mathbb{Q}$-factorial terminalization.

We have the following result

Theorem: There is a bijection between:

(i) G-equivariant covers $\widetilde{\mathbb{O}}$ of nilpotent orbits in $\mathfrak{g}^*$

(ii) Pairs $(L, \widetilde{\mathbb{O}}_L)$, where L is a Levi subgroup in G, $\widetilde{\mathbb{O}}_L$ is an L-equivariant cover of a nilpotent orbit in $[\mathfrak{l},\mathfrak{l}]^*$ (note that $[\mathfrak{l},\mathfrak{l}]$ is again a s/simple Lie algebra) s.t. X_L is $\mathbb{Q}$-factorial terminal. The pairs $(L,\tilde{\mathbb{O}}_L)$ are viewed up to suitably defined G-conjugacy.

(ii) to (i): we pick a parabolic P w. Levi L and consider $Y=\mu^{-1}(0)/P$. Then $\tilde{\mathbb{O}}$ is the unique open G-orbit in Y (only depending on L) & $Y\longrightarrow X=\operatorname{Spec}\mathbb{C}[\tilde{\mathbb{O}}]$ is a $\mathbb{Q}$-factorial terminalization. Finally, $\mathfrak{h}_X=H^2(Y^{reg},\mathbb{C})\xrightarrow{\sim}(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$.

In the next lecture we'll explain how to construct a quantization of $\mathbb{C}[\tilde{\mathbb{O}}]$ starting from a parameter $\lambda\in\mathfrak{h}_X$. We'll use "quantum Hamiltonian reduction".

Remarks: Let's comment on various parts of the proof of Thm.

(a) If X_L is $\mathbb{Q}$-factorial & terminal, then so is Y (exercise).

(b) $H^2(Y^{reg},\mathbb{C})\xrightarrow{\sim}(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ comes from the spectral sequence for the cohomology of the fiber bundle

$$Y^{reg}=G\times^P(X_L^{reg}\times(\mathfrak{g}/\mathfrak{p})^*)\longrightarrow G/P$$

thx to $H^i(X_L^{reg},\mathbb{C})\simeq H^i(\tilde{\mathbb{O}}_L,\mathbb{C})=\{0\}$ for $i=1,2$. The $i=1$ case holds always, & $i=2$ case holds b/c X_L is $\mathbb{Q}$-factorial.

(c) The claim that Y has an open G-orbit that is a cover of a nilpotent orbit is classical going back to Lusztig & Spaltenstein (1979). The proof uses a certain "Poisson deformation" of Y, it's sketched in the complement section.

(d) To prove the bijection claim one then needs to construct a map from (i) to (ii). This is based on looking at a Poisson deformation of X, essentially done by I.L. in 2016.

3) Complements.

3.1) Sketch of proof of (c)

We need to establish the following.

Lemma: $\dim \mu^{-1}(0)/P = \dim \operatorname{im} \mu'$, i.e. μ' is generically finite.

Sketch of proof: The proof is based on an idea fundamental for this subject – deformation. Set $\mathfrak{z} := (\mathfrak{l}^*)^L$. We have natural embeddings $\mathfrak{z} \hookrightarrow \mathfrak{l}^* \hookrightarrow \mathfrak{p}^*, \mathfrak{g}^*$ (for the embedding into $\mathfrak{g}^*$ take the unique L-equivariant embedding whose composition w. the restriction map $\mathfrak{g}^* \to \mathfrak{l}^*$ is the identity). The natural morphism $\mu^{-1}(\mathfrak{z}) \to \mathfrak{z}$ is P-invariant so descends to

$$\mu^{-1}(\mathfrak{z})/P \longrightarrow \mathfrak{z} \qquad (1)$$

We can identify $\mu^{-1}(\mathfrak{z})/P \simeq G \times^P (\mathfrak{z} \times X_L \times (\mathfrak{g}/\mathfrak{p})^*)$ and (1) becomes $[g, z, \alpha, x] \mapsto z$. In particular, all fibers of (1) have the same dimension.

We still have the G-equivariant map $\mu': \mu^{-1}(\mathfrak{z})/P \to \mathfrak{g}^*$.

Fact 1: For a Zariski generic $z \in \mathfrak{z}$, we have an isomorphism $G \times^P (\{z\} \times X_L \times (\mathfrak{g}/\mathfrak{p})^*) \xrightarrow{\sim} G \times^L (\{z\} \times X_L)$ and the morphism $\mu': G \times^P (\{z\} \times X_L \times (\mathfrak{g}/\mathfrak{p})^*) \longrightarrow \mathfrak{g}^*$ is finite.

Now consider $\mu': \mu^{-1}(\mathbb{C}z)/P \longrightarrow \mathfrak{g}^*$. We have

$$\dim \mu^{-1}(\mathbb{C}z)/P = \dim Y + 1.$$

By Fact 1, the image of $\mu^{-1}(\mathbb{C}z)/P$ in $\mathfrak{g}^*$ has dimension equal to $\dim \mu^{-1}(\mathbb{C}z)/P$. It remains to show that $\mu'(Y)$ is a divisor in $\mu'(\mu^{-1}(\mathbb{C}z)/P)$. For this, consider the composition of μ' w. the quotient morphism $\pi_G: \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/\!/G$. For any $z' \in \mathfrak{z}$ the image of $\mu^{-1}(z')/P$ under $\pi_G \circ \mu'$ is $\pi_G(z')$ (w. z' viewed as an element of $\mathfrak{g}^*$). We have $\pi_G(z') = 0 \Leftrightarrow z' = 0$. So $\pi_G \circ \mu'(\mu^{-1}(\mathbb{C}z)/P)$ is a curve and $\mu'(Y)$ is the preimage of a point in the curve under $\pi_G: \mu'(\mu^{-1}(\mathbb{C}z)/P) \to \mathfrak{g}^*/\!/G$. So $\operatorname{codim}_{\mu'(\mu^{-1}(\mathbb{C}z)/P)} \mu'(Y) \leq 1$, which finishes the proof $\square$

3.2) Construction of $\mathfrak{h}_X, W_X$ from the geometry of X.

Above we have explained how to recover the space $\mathfrak{h}_X$ from $Y \longrightarrow X$, a $\mathbb{Q}$-factorial terminalization. Now we'll explain how to recover $\mathfrak{h}_X$ & W_X from X itself, essentially due to Namikawa.

We start w. an important example.

Example: Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup. These subgroups $\overset{\sim}{\longleftrightarrow}$ type ADE diagrams. To recover the diagram from a subgroup one considers the minimal (symplectic) resolution $Y := \widetilde{\mathbb{C}^2/\Gamma} \longrightarrow X = \mathbb{C}^2/\Gamma$. The fiber over 0 is the union of $\mathbb{P}^1$'s. Two components either don't intersect or intersect transversally at a single point. So we can encode this as a graph: the vertices are the components and we have a non-oriented edge between two vertices if the components intersect. The resulting graph is an ADE Dynkin diagram and this classifies Γ up to conjugacy.

The space $H^2(Y, \mathbb{C})$ has basis labelled by the components of the zero fiber. Let $\mathfrak{h}_\Gamma$ denote the Cartan subalgebra in the simple Lie algebra of the same ADE type as Γ. We identify $H^2(Y, \mathbb{C}) \xrightarrow{\sim} \mathfrak{h}_\Gamma$ mapping the basis element in the l.h.s. to the simple coroot in the r.h.s.

Let's proceed to the general case. Here $\mathfrak{h}_X$ is the direct sum of several pieces: one corresponds to X^{reg}, equal to $H^2(X^{reg}, \mathbb{C})$ and the others are contributions of "codim 2 symplectic leaves" in X.

Definition: Let X be a Poisson variety. By a symplectic leaf we mean an irreducible local closed subvariety $\mathcal{L} \subset X$ s.t.

- $\mathcal{L}$ is smooth.
- $\mathcal{L}$ is a Poisson subvariety meaning that it's ideal is closed under taking the bracket w. $\mathcal{O}_X$.
- W.r.t. the induced Poisson structure on $\mathcal{O}_\mathcal{L}$, $\mathcal{L}$ is symplectic.

Example: Let $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit. Then the symplectic leaves in $\overline{\mathbb{O}}$ are exactly the G-orbits.

Assume until the end of the section that X is singular symplectic.

Fact (Kaledin): X has finitely many symplectic leaves and is, moreover, their disjoint union

Let $\mathcal{L}_1,\dots,\mathcal{L}_k$ be the codim 2 leaves of X so that $X^{reg}\sqcup\bigsqcup_{i=1}^{k}\mathcal{L}_i$ is the open subvariety in X whose complement has codim $\geqslant 4$. To each of them we will associate a vector space $\mathfrak{h}_i$ and a reflection group W_i acting on $\mathfrak{h}_i$ so that $\mathfrak{h}_X=\bigoplus_{i=0}^{k}\mathfrak{h}_i$ & $W_X=\prod_{i=1}^{k}W_i$ ($\mathfrak{h}_0:=H^2(X^{reg},\mathbb{C})$ w. trivial action).

We can consider the formal transversal slice Σ_i to $\mathcal{L}_i$. It has dimension 2 so must be the formal neighborhood of 0 in $\mathbb{C}^2/\Gamma_i$ for uniquely determined $\Gamma_i\subset SL_2(\mathbb{C})$ (there are no other 2-dimensional symplectic singularities). This gives the Cartan space $\mathfrak{h}_{\Gamma_i}$ and the Weyl group W_{Γ_i} of the same ADE type.

It turns out that the fundamental group $\pi_1(\mathcal{L}_i)$ acts on $\mathfrak{h}_{\Gamma_i}$, W_{Γ_i} in a compatible way, and $\mathfrak{h}_i=\mathfrak{h}_{\Gamma_i}^{\pi_1(\mathcal{L}_i)}$, $W_i=W_{\Gamma_i}^{\pi_1(\mathcal{L}_i)}$.

Let's explain how $\pi_1(\mathcal{L}_i)$ acts. The action comes from permuting the simple coroots in $\mathfrak{h}_{\Gamma_i}$. Consider a $\mathbb{Q}$-factorial terminalization $Y\longrightarrow X$. Its base change to Σ_i has to be a symplectic resolution $\widetilde{\Sigma}_i\to\Sigma_i$. So the fiber of $Y\to X$ over each point in $\mathcal{L}_i$ is the 0-fiber of $\widetilde{\mathbb{C}^2/\Gamma_i}\longrightarrow\mathbb{C}^2/\Gamma_i$. When a point travels around a loop, the components of the fiber may get permuted. This gives an action of $\pi_1(\mathcal{L}_i)$ on the Dynkin diagram associated to Γ_i. This gives the actions on $\mathfrak{h}_{\Gamma_i}, W_{\Gamma_i}$ that we need.

Example: Let $\mathfrak{g}$ be a simple Lie algebra and $X=\mathcal{N}$. We have $\mathfrak{h}_0=\{0\}$. There is one codim 2 leaf (=orbit) known as the subregular orbit. For classical Lie algebras, they correspond to the following partitions:

- $\mathfrak{sl}_n$: $(n-1,1)$
- $\mathfrak{so}_{2n+1}$ $(2n-1,1^2)$
- $\mathfrak{sp}_{2n}$ $(2n-2,2)$
- $\mathfrak{so}_{2n}$ $(2n-3,3)$

If $\mathfrak{g}$ is simply laced, then the slice to the subregular orbit is of the same type as $\mathfrak{g}$ and the monodromy action is trivial. The types of slices for non-simply laced Lie algebras together w. $\pi_1(\mathcal{L})$ are in the following table:

$\mathfrak{g}$	Σ	$\pi_1(\mathcal{L})$
B_n	A_{2n-1}	$\mathbb{Z}/2\mathbb{Z}$
C_n	D_{n+1}	$\mathbb{Z}/2\mathbb{Z}$
F_4	E_6	$\mathbb{Z}/2\mathbb{Z}$
G_2	D_4	S_3

The group $\pi_1(\mathcal{L})$ acts by diagram foldings.

In all cases we recover the description of $\mathfrak{h}_{\mathcal{N}}$ as $\mathfrak{h}^*$, see Example in Section 1.