Quantized symplectic singularities & applications to Lie theory, Lec 4.

1) Quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$
2) Harish-Chandra bimodules.
3) Complements.

1.0) Recap.

Let G be a semisimple algebraic group, $\mathfrak{g}$ its Lie algebra, $\mathbb{O}$ a nilpotent orbit in $\mathfrak{g}$ & $\tilde{\mathbb{O}}$ a G-equivariant cover of $\mathbb{O}$. Let $A:=\mathbb{C}[\tilde{\mathbb{O}}]$, $X=\operatorname{Spec} A$. In Section 3 of Lec 2, we have stated that the filtered quantizations are classified by the points of $\mathfrak{h}_X/W_X$, where $\mathfrak{h}_X$ is a finite dimensional vector space and W_X is a crystallographic reflection group. We have explained how to compute $\mathfrak{h}_X$: $\mathfrak{h}_X=H^2(Y^{reg},\mathbb{C})$, where Y is a $\mathbb{Q}$-factorial terminalization of X, Section 1.2 of Lec 3. According to Sec 2 of Lec 3, Y has the following form. Pick a Levi subgroup $L\subset G$, and an L-equivariant cover $\tilde{\mathbb{O}}_L$ of a nilpotent orbit in $\mathfrak{l}^*$. Suppose $X_L:=\operatorname{Spec}\mathbb{C}[\tilde{\mathbb{O}}_L]$ is $\mathbb{Q}$-factorial terminal.

Let P be a parabolic subgroup of G w. Levi L. Consider the Hamiltonian action of P on $T^*G\times X_L$, $p(g,\alpha,x)=(gp^{-1}, p\alpha, px)$. The moment map is $\mu: T^*G\times X_L\longrightarrow \mathfrak{p}^*$, $(g,\alpha,x)\mapsto -\alpha|_{\mathfrak{p}}+\mu_L(x)$. Then $Y=\mu^{-1}(0)/P\simeq G\times^P(X_L\times(\mathfrak{g}/\mathfrak{p})^*)$. This is a $\mathbb{Q}$-factorial terminalization of $X=\operatorname{Spec}\mathbb{C}[\tilde{\mathbb{O}}]$, where $\tilde{\mathbb{O}}\subset Y$ is the open G-orbit (depending only on $L,\tilde{\mathbb{O}}_L$). We have $\mathfrak{h}_X=H^2(Y^{reg},\mathbb{C})=(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$.

An important remark is in order. As was discussed in Sec 1 of Lec 2, $\mathbb{C}^\times \curvearrowright X_L$ rescaling the Poisson bracket by $t\mapsto t^{-d}$ for some $d\in\mathbb{Z}_{>0}$. Consider the action of $\mathbb{C}^\times$ on $T^*G\times X_L$ by $t.(g,\alpha,x)= =(g,t^{-d}\alpha, t.x)$. It descends to $Y=\mu^{-1}(0)/P$ & rescales $\{\cdot,\cdot\}$ on $\mathcal{O}_Y$ by $t\mapsto t^{-d}$.

1.1) Quantization of Y.

Let $\eta: Y=G\times^P(X_L\times(\mathfrak{g}/\mathfrak{p})^*)\longrightarrow G/P$ denote the projection, it's $\mathbb{C}^\times$-invariant. So $\eta_*\mathcal{O}_Y$ becomes the sheaf of (positively) graded Poisson algebras on G/P. We can talk about its filtered quantizations: quasicoherent sheaves $\mathcal{D}$ of $\mathcal{O}_{G/P}$-modules equipped w.

- an associative $\mathbb{C}$-algebra structure,
- a filtration $\mathcal{D}=\bigcup_{i\geqslant 0}\mathcal{D}_{\leqslant i}$ by $\mathcal{O}_{G/P}$-submodules satisfying $[\mathcal{D}_{\leqslant i},\mathcal{D}_{\leqslant j}]\subset\mathcal{D}_{\leqslant i+j-d}\rightsquigarrow\{\cdot,\cdot\}$ on $\operatorname{gr}\mathcal{D}$
- and an $\mathcal{O}_{G/P}$-linear isomorphism $\operatorname{gr}\mathcal{D}\xrightarrow{\sim}\eta_*\mathcal{O}_Y$ of graded Poisson algebras.

Goal: for $\lambda\in(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ produce a filtered quantization $\mathcal{D}_\lambda$ of $\eta_*\mathcal{O}_{G/P}$.

For this we "quantize" the construction of Y.

• As was mentioned in Sec 2 of Lec 3, $H^2(X_L^{reg};\mathbb{C})=\{0\}$. So by Theorem in Sec 3 of Lec 2, $\mathbb{C}[X_L]$ admits a <u>unique</u> filtered quantization, to be denoted by $\mathcal{A}_L$. A quantum counterpart of $\mathbb{C}[T^*G]\otimes\mathbb{C}[X_L]$ is $\mathcal{D}(G)\otimes\mathcal{A}_L$.

- We have the classical comoment map

$$\varphi:\mathfrak{p}\longrightarrow\mathbb{C}[T^*G]\otimes\mathbb{C}[X_L]$$

$\varphi(\xi)=-\xi_r\otimes 1+1\otimes\varphi_L(\xi)$, where ξ_r is the left-invariant vector field on G corresponding to ξ (and viewed as a fiber-wise linear function on T^*G, and $\varphi_L:\mathfrak{p}\to\mathbb{C}[X_L]$ is the comoment map – dual to $X_L\to\mathfrak{l}^*\hookrightarrow\mathfrak{p}^*$. Note that $\varphi(\xi)$ is a homogeneous deg d element.

We need a quantum counterpart of φ.

Definition: Let $\mathcal{A}$ be an associative algebra w. a rational action of an algebraic group R by algebra automorphisms. By a quantum comoment map for this action we mean an R-equivariant linear map $\Phi:\mathfrak{r}\to\mathcal{A}$ s.t. $[\Phi(\xi),\cdot]=\xi_{\mathcal{A}}\ \forall\,\xi\in\mathfrak{r}$.

Example: $\xi\mapsto-\xi_r$ (resp. $\xi\mapsto\xi_\ell$) $:\mathfrak{g}\to\mathcal{D}(G)$ is a quantum comoment map for $G\curvearrowright\mathcal{D}(G)$ induced by $G\curvearrowright G$ from the right (resp. left).

Exercise 1: Assume $\tilde{\mathbb{O}}_L$ is an arbitrary L-equivariant cover of a nilpotent L-orbit. Show that $\varphi_L:\mathfrak{l}\to\mathbb{C}[X_L]_d$ lifts to a Lie algebra homomorphism $\Phi_L:\mathfrak{l}\to\mathcal{A}_{L,\leq d}$ (meaning that Φ_L mod $\mathcal{A}_{L,\leq d-1}=\varphi$). Moreover, $\exists!$ lift that vanishes on $\mathfrak{z}(\mathfrak{l})$. Finally, L acts on $\mathcal{A}_L$ by filtered algebra homomorphisms so that Φ_L is a quantum comoment map.

Take this lift and inflate it to $\varphi_L : \mathfrak{p} \to \mathcal{A}_L$.

Now we are ready to define $\varphi_\lambda : \mathfrak{p} \to D(G) \otimes \mathcal{A}_L$ for $\lambda \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. Let $\rho_{G/P}$ denote $\frac{1}{2}$(the character of $\mathfrak{l}$ in $\Lambda^{top}(\mathfrak{g}/\mathfrak{p})$). We can view $\lambda, \rho_{G/P}$ as characters of $\mathfrak{p}$ via $\mathfrak{p} \twoheadrightarrow \mathfrak{l}$. Set

$$\varphi_\lambda(\xi) = -\xi_r \otimes 1 + 1 \otimes \varphi_L(\xi) - \langle \lambda - \rho_{G/P}, \xi \rangle.$$

This a quantum comoment map.

- To get a quantization of Y we perform the quantum Hamiltonian reduction.

Exercise 2: Let $R, \mathcal{A}, \varphi$ have the same meaning as in the definition above. Show that $[\mathcal{A}/\mathcal{A}\varphi(\mathfrak{r})]^R$ has a unique associative algebra structure s.t.

$$(a + \mathcal{A}\varphi(\mathfrak{r})) \cdot (b + \mathcal{A}\varphi(\mathfrak{r})) = ab + \mathcal{A}\varphi(\mathfrak{r})$$

This algebra is known as the quantum Hamiltonian reduction.

Remark: Note that if $\mathcal{A}$ is filtered w. $\deg[\cdot,\cdot] \leq -d$ ($[\mathcal{A}_{\leq i}, \mathcal{A}_{\leq j}] \subset \mathcal{A}_{\leq i+j-d}\ \forall i,j$) & $\operatorname{im}\varphi \subset \mathcal{A}_{\leq d}$, then $[\mathcal{A}/\mathcal{A}\varphi(\mathfrak{r})]^R$ inherits a filtration from $\mathcal{A}$ & $\deg[\cdot,\cdot] \leq -d$.

Apply this to our situation. Let $\mathcal{A} = D(G) \otimes \mathcal{A}_L$, we can view it as a P-equivariant quasicoherent sheaf on G. So is $\mathcal{A}/\mathcal{A}\varphi_\lambda(\mathfrak{p})$. Recall the projection $\omega : G \to G/P$ and set

$$\mathcal{D}_\lambda := [\omega_*(\mathcal{A}/\mathcal{A}\varphi_\lambda(\mathfrak{p}))]^P.$$

Using the important exercise we equip $\mathcal{D}_\lambda$ (sheaf of) algebra structure. It's filtered by Remark after the exercise. We will elaborate on this and a proof of the fact below in the complement section.

Fact: $\mathcal{D}_\lambda$ is a filtered quantization of $\eta_* \mathcal{O}_Y$.

Example: Let $X = \mathcal{N}$, $Y = T^*(G/B)$ (so $L = T$, $P = B$, $X_L = \{0\}$). In this case $\rho_{G/B}$ is the usual ρ and $\mathcal{D}_\lambda = \mathcal{D}^{\lambda-\rho}_{G/B}$, the sheaf of $(\lambda-\rho)$-twisted differential operators. More generally, we get twisted diff. operators in the case when $Y = T^*(G/P)$.

Remarks: • In fact, all filtered quantizations of $\eta_* \mathcal{O}_Y$ are of the form $\mathcal{D}_\lambda$, and $\mathcal{D}_\lambda \not\cong \mathcal{D}_{\lambda'}$ for $\lambda \neq \lambda'$. We'll comment on this in the complement section.

• One could (and should) ask what $\mathcal{A}_L$ looks like. To an extent, this is addressed in the next section.

1.2) Quantizations of $\mathbb{C}[X]$.

Proposition: $\mathcal{A}_\lambda := \Gamma(\mathcal{D}_\lambda)$ is a filtered quantization of $\mathbb{C}[X]$.

Sketch of proof: this is a formal consequence of

(1) $\operatorname{gr} \mathcal{D}_\lambda = \eta_* \mathcal{O}_Y$

(2) $\Gamma(G/P, \eta_* \mathcal{O}_Y) (= \mathbb{C}[Y]) \xleftarrow{\sim} \mathbb{C}[X]$,

(3) $H^1(G/P, \eta_* \mathcal{O}_Y)(= H^1(Y, \mathcal{O}_Y)) = 0$.

(1) is Fact in Sec 1.1; (2) & (3) follows from the following algebro-geometric fact: if X is singular symplectic, Y is normal & Poisson w. a proper birational morphism $\pi: Y \to X$, then $\pi_* \mathcal{O}_Y \xleftarrow{\sim} \mathcal{O}_X$ & $R^i\pi_* \mathcal{O}_Y = 0$ $\forall i > 0$ (the latter follows from symplectic singularities being "rational" - shown by Beauville). Using this vanishing one checks that $gr\Gamma(\mathcal{D}_\lambda) \xrightarrow{\sim} \Gamma(gr\mathcal{D}_\lambda)$, which then implie $\Gamma(\mathcal{D}_\lambda)$ is a filtered quantization of $\mathbb{C}[Y] = \mathbb{C}[X]$.

Let's explain how $gr\,\Gamma(\mathcal{D}_\lambda) \xrightarrow{\sim} \Gamma(gr\,\mathcal{D}_\lambda)$ follows: $gr\,\mathcal{D}_\lambda = \eta_* \mathcal{O}_Y$ gives SES's: $0 \to \mathcal{D}_{\lambda, \leq i-1} \to \mathcal{D}_{\lambda, \leq i} \to (\eta_* \mathcal{O}_Y)_i \to 0$, $\forall i \geq 0$. We know $H^1(G/P, (\eta_* \mathcal{O}_Y)_i) = 0$ $\forall i \geq 0$. By induction, $H^1(G/P, \mathcal{D}_{\lambda, \leq i-1}) = 0$ $\forall i \rightsquigarrow$ SES $0 \to \Gamma(\mathcal{D}_{\lambda, \leq i-1}) \to \Gamma(\mathcal{D}_{\lambda, \leq i}) \to \Gamma((\eta_* \mathcal{O}_Y)_i) \to 0$ $\Leftrightarrow$ $gr\,\Gamma(\mathcal{D}_\lambda) \xrightarrow{\sim} \Gamma(gr\,\mathcal{D}_\lambda)$. $\square$

Example: For $X = \mathcal{N}$, $Y = T^*(G/B)$, it's classically known that $\Gamma(\mathcal{D}^{\lambda-\rho}_{G/B}) = \mathcal{U}_\lambda$ $(= \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{m}_\lambda)$ from Sec 3 of Lec 2.

Remarks:

I) $\mathcal{D}_\lambda$'s are pairwise distinct as quantizations, but $\mathcal{A}_\lambda$'s aren't. First, one can determine when $\mathcal{A}_\lambda \simeq \mathcal{A}_{\lambda'}$, a G-equivariant filted algebra isomorphism.

Consider $N_G(L) \subset G$. This group acts on L and hence on L-equivariant covers of nilpotent orbits (by twisting the L-action - and hence the moment map). So it makes sense to speak about the stabilizer of $\widetilde{\mathbb{O}}_L$ under this action (if $\widetilde{\mathbb{O}}_L \subset \mathfrak{l}^*$;

then this is just all elements of $N_G(L)$ that preserve $\widetilde{\mathbb{O}}_L$ as a subset). Denote this subgroup of $N_G(L)$ by $N_G(L,\widetilde{\mathbb{O}}_L)$. We have $L \trianglelefteq N_G(L,\widetilde{\mathbb{O}}_L)$ & $N_G(L,\widetilde{\mathbb{O}}_L)/L \curvearrowright (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$.

Claim (basically, I.L. 16): $\mathcal{A}_\lambda \simeq \mathcal{A}_{\lambda'}$ as filtered algebras $\Leftrightarrow$ λ, λ' are in the same $N_G(L,\widetilde{\mathbb{O}}_L)$-orbit.

Comment: Y depends on the choice of P & so does $\mathcal{D}_\lambda$ but one can show that $\mathcal{A}_\lambda$ doesn't. Let use P as a superscript to indicate the dependence on P: $Y^P, \mathcal{D}_\lambda^P$. Then n gives rise to $Y^P \xrightarrow{\sim} Y^{nP}$, $\mathcal{D}_\lambda^P \xrightarrow{\sim} \mathcal{D}_{n\lambda}^{nP} \rightsquigarrow \mathcal{A}_\lambda = \Gamma(\mathcal{D}_\lambda^P) \xrightarrow{\sim} \Gamma(\mathcal{D}_{n\lambda}^{nP}) = \mathcal{A}_{n\lambda}$. This proves $\Leftarrow$ in the proposition $\square$

One can use the claim & the comment to describe W_X and hence to answer when $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$ as quantizations. Note that a filtered algebra isomorphism $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$ gives a Poisson graded algebra automorphism of $\mathbb{C}[X]$. These automorphisms form a group that can be shown to coincide w. the group $\mathrm{Aut}_G(\widetilde{\mathbb{O}})$ of G-equivariant symplectomorphisms of $\widetilde{\mathbb{O}}$, it's finite. So we get a group homomorphism $N_G(L,\widetilde{\mathbb{O}}_L) \longrightarrow \mathrm{Aut}_G(\widetilde{\mathbb{O}})$.

Fact 2 (I.L., Namikawa) We have SES

$$1 \longrightarrow W_X \longrightarrow N_G(L,\widetilde{\mathbb{O}}_L)/L \longrightarrow \mathrm{Aut}_G(\widetilde{\mathbb{O}}) \longrightarrow 1.$$

For example when $\widetilde{\mathbb{O}} \xrightarrow{\sim} \mathbb{O} \subset \mathfrak{g}^*$, then $\mathrm{Aut}_G(\mathbb{O}) = 1$.

II) Using this description (and some more) we can produce an algebraic version of Orbit method, essentially as conjectured by Vogan in the 90's.

Thm (LMBM'21): There's a natural bijection between:

1) Filtered quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$ for all equivariant covers $\tilde{\mathbb{O}}$ of <u>nilpotent</u> orbits, up to filtered algebra iso.

2) All equivariant covers of all (co)adjoint G-orbits.

Under this correspondence, the cover $\tilde{\mathbb{O}}$ of a nilpotent orbit (in 2)) goes to the quantization $\mathcal{A}_0$ of $\mathbb{C}[\tilde{\mathbb{O}}]$, the canonical quantization.

III) Can we describe $\mathcal{A}_\lambda$ "explicitly"? We can e.g. when $\lambda=0$ & $\tilde{\mathbb{O}}\subset\mathfrak{g}^*$. By Exercise 1 (also can be seen by the construction), we have the unique quantum comoment map $\varphi_G: U(\mathfrak{g})\longrightarrow\mathcal{A}_0$. The following result requires quite a lot of work (and describes $\mathcal{A}_0$ as an algebra w/o filtration).

Thm (LMBM & MBM): • $\mathcal{A}_0$ is a simple algebra

• $\ker\varphi_G$ is a maximal ideal (that we can recover starting from $\tilde{\mathbb{O}}$)

• If $\tilde{\mathbb{O}}\xrightarrow{\sim}\mathbb{O}(\subset\mathfrak{g}^*)$, then $\operatorname{im}\varphi=\mathcal{A}_0$ (more generally, if $\tilde{\mathbb{O}}/\operatorname{Aut}_G(\tilde{\mathbb{O}})\xrightarrow{\sim}\tilde{\mathbb{O}}$, then $\operatorname{im}\varphi=\mathcal{A}_0^{\operatorname{Aut}_G(\tilde{\mathbb{O}})}$.

2) Harish-Chandra bimodules.

Definition (classical): A HC $\mathcal{U}(\mathfrak{g})$-bimodule is a finitely generated $\mathcal{U}(\mathfrak{g})$-bimodule $\mathcal{B}$ that is "$\mathrm{ad}(\mathfrak{g})$-locally finite": $\forall\, b\in\mathcal{B}\ \exists$ fin. dim'l $\mathrm{ad}(\mathfrak{g})$-stable subspace $\mathcal{B}_0\subset\mathcal{B}$ w. $b\in\mathcal{B}_0$.

Example: • $\mathcal{U}(\mathfrak{g})$ is HC bimodule.
• Every sub- & quotient bimodule of a HC bimodule is HC.

Exercise: • Let V be a finite dimensional $\mathfrak{g}$-rep'n. Show that $V\otimes\mathcal{U}(\mathfrak{g})$ is a HC bimodule w.r.t. $(v\otimes a)\xi := v\otimes a\xi$, $\xi(v\otimes a):=\xi v\otimes a+v\otimes\xi a$, $v\in V$, $a\in\mathcal{U}(\mathfrak{g})$, $\xi\in\mathfrak{g}$.
• Moreover, every HC bimodule is a quotient of some $V\otimes\mathcal{U}(\mathfrak{g})$.

Let's explain why Harish-Chandra cared: he wanted to have algebraic counterparts of unitary rep's. For simplicity, assume G is simply connected. Let H be a unitary G-representation (some kind of L^2-space). Inside, there's the "C^∞-part", $C^\infty(H)$, it now carries a $\mathfrak{g}$-action, by skew-Hermitian operators.

Let $K\subset G$ be a max'l compact subgroup. Consider the "K-finite part" $C^\infty(H)_{K\text{-fin}}$ consisting of all vectors lying in K-stable fin. dim'l subspaces. This is a complex vector space w. $\mathfrak{g}$-action. If a <u>real</u> Lie algebra (resp. algebraic group) acts on a <u>complex</u> vector space, then the action extends to the complexification. So $(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}, K_{\mathbb{C}})$ act on $C^\infty(H)_{K\text{-fin}}$ act (compatibly). Of course, $\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\simeq\mathfrak{g}\oplus\mathfrak{g}$ & $\mathrm{Lie}(K_{\mathbb{C}})=\mathfrak{g}$ embedded into $\mathfrak{g}\oplus\mathfrak{g}$ diagonally.

A $\mathfrak{g}\oplus\mathfrak{g}$-module is the same thing as a $\mathcal{U}(\mathfrak{g})$-module. The action of the diagonal copy of $\mathfrak{g}$ becomes the adjoint action. So $C^\infty(\mathcal{H})_{K\text{-fin}}$ becomes a $\mathcal{U}(\mathfrak{g})$-bimodule w. locally finite $\operatorname{ad}(\mathfrak{g})$-action.

Thm (Harish-Chandra): $\mathcal{H}\mapsto C^\infty(\mathcal{H})_{K\text{-fin}}$ defines a bijection between:

- Unitary irreps $\mathcal{H}$ of G
- Irreducible HC bimodules that are "unitarizable": have a positive definite Hermitian form w. certain invariance property (saying that $\mathfrak{g}\subset\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\simeq\mathfrak{g}\oplus\mathfrak{g}$ acts by skew-Hermitian operators).

While this "algebrizes" the problem of classifying unitary G-irreps, the unitarizability condition is still very hard to check. Experimental evidence suggests that the class of unitarizable HC $\mathcal{U}(\mathfrak{g})$-modules has "big intersection" with HC bimodules over quantizations of $\mathbb{C}[\widetilde{\mathbb{O}}]$'s. We will discuss those in the final lecture.

3) Complements

3.1) Comments on the classification of quantizations of $\eta_*\mathcal{O}_Y$.

- Why $\mathcal{D}_\lambda$ is a filtered quantization of $\eta_*\mathcal{O}_Y$. In general, $\varphi=\Phi+\mathcal{A}_{\leq d-1}$ gives an iso of graded quasi-coherent sheaves on G/P:

$$\mathcal{O}_{\mu^{-1}(0)}\twoheadrightarrow \operatorname{gr}[(\mathcal{D}(G)\otimes\mathcal{A}_L)/(\mathcal{D}(G)\otimes\mathcal{A}_L)\operatorname{im}\varphi_\lambda] \qquad (1)$$

This epimorphism is an iso. A basic reason for this is that P acts on $\mu^{-1}(0)$ freely (no stabilizers). From here one deduces that, for a basis $\xi_1,\dots,\xi_n$ of $\mathfrak{p}$,

(*) the elements $\varphi(\xi_1),\dots,\varphi(\xi_n)$ form a regular sequence

this can be also seen directly: $\operatorname{codim}_{T^*G\times X_L} \mu^{-1}(0)=\dim\mathfrak{p}$

Using (*) and some Commutative algebra (regular $\Rightarrow$ the 1st Kostul homology group vanishes) one can show that (1) is an isomorphism. Passing to the P-invariants is still an isomorphism - also follows from the freeness.

• Why $\mathcal{D}_\lambda \simeq \mathcal{D}_{\lambda'} \Rightarrow \lambda\simeq\lambda'$ and $\mathcal{D}_\lambda$'s exhaust all quantizations of $\eta_*\mathcal{O}_Y$.

Easy case: $Y=T^*(G/P)$. Here we recover the classification of sheaves of twisted differential operators.

The general case: $Y=G\times^P((\mathfrak{g}/\mathfrak{p})^*\times X_L)$ is very mildly singular. It follows from the work of Bezrukavnikov-Kaledin & I.L. that the filtered quantizations of $\eta_*\mathcal{O}_Y$ are classified by $H^2(Y^{reg};\mathbb{C})\ (=(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*)$ by means of the so called period map. One can prove that the period of $\mathcal{D}_\lambda$ is λ, (I.L. 2010).

3.2) Barbasch-Vogan constr'n & glimpses of symplectic duality

Here we are concerned with understanding the kernels of the quantum comoment maps $\mathcal{U}(\mathfrak{g})\longrightarrow\mathcal{A}_0$, where $\mathcal{A}_0$ is the canonical (parameter 0) quantization of some $\mathbb{C}[\tilde{\mathbb{O}}]$. It turns out that at least some of them have "meaning" & have appeared before.

In the study of unitary representations of real Lie groups there's an important - yet still conjectural - class of representations called unipotent. Under the (non-existing) Orbit method correspondence those are unitary irreps that correspond to nilpotent orbits. A formal definition for HC bimodules will be suggested in the next lecture.

In 85, Barbasch & Vogan proposed a partial definition: special unipotent representations. The first step is to define a family of ideals in $\mathcal{U}(\mathfrak{g})$. To describe their construction we need to describe the maximal (w.r.t. $\subset$) 2-sided ideals. Recall that we write $\mathcal{Z}$ for the center of $\mathcal{U}(\mathfrak{g})$. Recall the identification $\mathcal{Z} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$.

Proposition: For every maximal ideal $I \subset \mathcal{U}(\mathfrak{g})$, we have $\operatorname{codim}_{\mathcal{Z}} I \cap \mathcal{Z} = 1$, so I defines a point in $\mathfrak{h}^*/W$. The resulting map $\{\text{max. 2-sided ideals in } \mathcal{U}(\mathfrak{g})\} \xrightarrow{\sim} \mathfrak{h}^*/W$ is a bijection.

Notation: For $\lambda \in \mathfrak{h}^*/W$, let $I_{max}(\lambda)$ denote the maximal ideal in $\mathcal{U}(\mathfrak{g})$ corresponding to λ under the bijection from the proposition.

Verma w. h.wt. $-\rho$

Example: $I_{max}(\rho) = \mathfrak{g}\,\mathcal{U}(\mathfrak{g})$, $I_{max}(0) = \operatorname{Ann}_{\mathcal{U}(\mathfrak{g})}(\Delta(-\rho))$.

Now we can introduce to Barbasch-Vogan ideals. We write $\mathfrak{g}^\vee$ for the Langlands dual Lie algebra (e.g. for $\mathfrak{g}=\mathfrak{so}_{2n+1}$ we get $\mathfrak{g}^\vee=\mathfrak{sp}_{2n}$). Let $\mathbb{O}^\vee$ be a nilpotent orbit in $\mathfrak{g}^\vee$. Pick $e^\vee\in\mathbb{O}^\vee$ and include it into an $\mathfrak{sl}_2$-triple $(e^\vee,h^\vee,f^\vee)$. Conjugating we can assume that $h^\vee\in\mathfrak{h}^\vee(=\mathfrak{h}^*)$. The element $h^\vee$ is defined up to conjugacy in W.

Definition: The special unipotent ideal $I_{\mathbb{O}^\vee}:=I_{max}(\frac{1}{2}h^\vee)$

Theorem (LMBM'21) $I_{\mathbb{O}^\vee}$ is the kernel of $\mathcal{U}(\mathfrak{g})\longrightarrow\mathcal{A}_0$ for a cover $\widetilde{\mathbb{O}}$ recovered from $\mathbb{O}^\vee$; we'll write $\tilde{d}(\mathbb{O}^\vee)$ for $\widetilde{\mathbb{O}}$.

To construct $\tilde{d}$ we need the following observation. Note that every 2-sided ideal $I\subset\mathcal{U}(\mathfrak{g})$ defines a $G\times\mathbb{C}^\times$-stable subvariety in $\mathfrak{g}^*$ (where $\mathbb{C}^\times$ acts by dilations) - the variety of 0's of $\operatorname{gr} I$, where gr is taken for the PBW filtration. Denote this subvariety by $V(I)$. If $\operatorname{codim}_Z Z\cap I<\infty$, then $V(I)\subset\mathcal{N}$ and if I is maximal (more generally, "primitive"), then $V(I)$ is irreducible.

Definition: The BV dual orbit $d(\mathbb{O}^\vee)$ is the unique open orbit in $V(I_{\mathbb{O}^\vee})$.

Examples: 1) $\mathfrak{g}=\mathfrak{sl}_n=\mathfrak{g}^\vee$. If $\mathbb{O}^\vee$ corresponds to a partition μ of

n, then $d(\mathbb{O}^\vee)$ corresponds to μ^t.

2) $\mathfrak{g} = \mathfrak{sp}_4 = \mathfrak{g}^\vee$. The following illustrates how d works:

$$\begin{array}{ccc} \mathbb{O}^\vee & & d(\mathbb{O}^\vee) \\ \{0\} & \longrightarrow & (4) \\ (2,1,1) & \longrightarrow & (2,2) \\ (2,2) & \nearrow & (2,1,1) \\ (4) & \longrightarrow & \{0\} \end{array}$$

Construction of $\tilde{d}(\mathbb{O}^\vee)$:

Case 1: $e^\vee$ is "distinguished" (not contained in a proper Levi). Then $\tilde{d}(\mathbb{O}^\vee)$ is the universal $Ad(\mathfrak{g})$-equivariant cover of $d(\mathbb{O})$. It turns that $\operatorname{Spec} \mathbb{C}[\tilde{d}(\mathbb{O}^\vee)]$ is $\mathbb{Q}$-factorial & terminal.

Case 2: general. Let $\mathfrak{l}^\vee$ be a minimal Levi subalgebra of $\mathfrak{g}^\vee$ containing $e^\vee$. Let $\mathfrak{l}$ be a corresponding Levi subalgebra of $\mathfrak{g}$ & $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}$ be a parabolic. Let $X_2 = \operatorname{Spec} \mathbb{C}[\tilde{d}(\mathbb{O}_2)]$, where $\tilde{d}(\mathbb{O}_2)$ is constructed as in Case 1. Then $\tilde{d}(\mathbb{O})$ is the open orbit in $G \times^P (X_2 \times (\mathfrak{g}/\mathfrak{p})^*)$.

Examples: • For $\mathfrak{g} = \mathfrak{sl}_n$, we have $\tilde{d}(\mathbb{O}) = d(\mathbb{O})$.

• For $\mathfrak{g} = \mathfrak{sp}_4$, $\tilde{d}$ sends the orbit $(2,1,1)$ to the orbit $(2,2)$ & $(2,2)$ to the double cover of $(2,2)$.

An inspiration for defining $\mathcal{I}$ comes from Symplectic duality predicted by Braden-Licata-Proudfoot-Webster and since then rigorously defined in some settings but not in ours. In the first approximation, this is a duality between conical singular symplectic varieties $X, X^\vee$ (w. some "decorations") that swaps certain invariants. The pair of varieties in our case is as follows:

- $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathcal{I}}(\mathbb{O}^\vee)]$
- $X^\vee$ = the intersection of the "Slodowy slice", $e^\vee + \mathfrak{z}_{\mathfrak{g}^\vee}(f^\vee)$, w. the nilpotent cone $\mathcal{N}^\vee \subset \mathfrak{g}^\vee$. The Slodowy slice is a transverse slice to $\mathbb{O}^\vee$ in $\mathfrak{g}^\vee$.