Quantized symplectic singularities & applications to Lie theory, Lec 5.

1) Harish-Chandra bimodules
2) Classification, application & generalization.

1.1) Definition

Let A be a positively graded ($\mathbb{Z}_{\geqslant 0}$-graded w. $A_0 = \mathbb{C}$) commutative algebra equipped w. a Poisson bracket of deg $-d$. Let $\mathcal{A}$ be a filtered quantization of A: a filtered associative algebra graded Poisson algebra isomorphism $\operatorname{gr}\mathcal{A} \xrightarrow{\sim} A$.

Definition: Let $\mathcal{B}$ be an $\mathcal{A}$-bimodule.

1) By a good filtration on $\mathcal{B}$ we mean an ascending vector space filtration $\mathcal{B} = \bigcup_{j\in\mathbb{Z}} \mathcal{B}_{\leqslant j}$ with the following properties:

(i) $\mathcal{B}_{\leqslant j} = \{0\}$ for some j.

(ii) $\mathcal{A}_{\leqslant i}\mathcal{B}_{\leqslant j}, \mathcal{B}_{\leqslant j}\mathcal{A}_{\leqslant i} \subset \mathcal{B}_{\leqslant i+j}$ $\forall i,j$. This shows, in particular, that $\operatorname{gr}\mathcal{B}$ is an A-bimodule.

(iii) $[\mathcal{A}_{\leqslant i}, \mathcal{B}_{\leqslant j}] \subset \mathcal{B}_{\leqslant i+j-d}$ $\forall i,j$. This shows that the left & right A-actions on $\operatorname{gr}\mathcal{B}$ coincide, so $\operatorname{gr}\mathcal{B}$ is A-module.

(iv) $\operatorname{gr}\mathcal{B}$ is a finitely generated A-module.

2) $\mathcal{B}$ is Harish-Chandra (shortly, HC) if it admits a good filtration.

Examples: • The regular bimodule $\mathcal{A}$ is HC.
• If $\mathcal{B}$ is HC, then all of its subs & quotients are.

Remarks: 1) A good filtration is non-unique. Nevertheless, the following is true: if $\mathcal{B}=\bigcup_j \mathcal{B}_{\leq j}=\bigcup_j \mathcal{B}'_{\leq j}$ are two good filtrations, then $\exists\, m_1, m_2\in\mathbb{Z}$ s.t. $\mathcal{B}_{\leq j-m_1}\subset \mathcal{B}'_{\leq j}\subset \mathcal{B}_{\leq j+m_2}\ \forall j$. Left as exercise.

2) For $\mathcal{A}=U(\mathfrak{g})$ w. PBW filtration, the two definitions of being HC (above & in Sec 2 of Lec 4) agree. Compare to the proof of Lemma below.

1.2) HC bimodules over quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$.

Now suppose that $\tilde{\mathbb{O}}$ is a G-equivariant cover of a nilpotent orbit in $\mathfrak{g}$. Let $\mathcal{A}$ be a filtered quantization of $\mathbb{C}[\tilde{\mathbb{O}}]$. Recall, Exer 1 in Sec 1.1 of Lec 4, that we have a quantum comoment map $\Phi: U(\mathfrak{g})\longrightarrow\mathcal{A}$. So every $\mathcal{A}$-bimodule can be viewed as a $U(\mathfrak{g})$-bimodule.

Lemma: If $\mathcal{B}$ is a HC $\mathcal{A}$-bimodule, then it's also HC when viewed as a $U(\mathfrak{g})$-bimodule.

Proof: We have $\Phi(\mathfrak{g})\subset\mathcal{A}_{\leq d}$. By (iii), $[\operatorname{im}\Phi, \mathcal{B}_{\leq j}]\subset\mathcal{B}_{\leq j}$. Since A is positively graded & $\operatorname{gr}\mathcal{B}$ is finitely generated over A, we see that $\dim(\operatorname{gr}\mathcal{B})_j<\infty$. By (i), $\dim\mathcal{B}_{\leq j}<\infty\ \forall j$. So every element in $\mathcal{B}$ is contained in a finite dimensional $\mathfrak{g}$-stable subspace. Also (iv) implies $\mathcal{B}$ is finitely generated left $\mathcal{A}$-module. Since $\mathbb{C}[\tilde{\mathbb{O}}]$ is finite over $\mathbb{C}[\mathfrak{g}]$, we see that $\mathcal{A}$ is a finitely generated left $\mathcal{U}(\mathfrak{g})$-module. It follows that $\mathcal{B}$ is a finitely generated left $\mathcal{U}(\mathfrak{g})$-module, hence a HC $\mathcal{U}(\mathfrak{g})$-bimodule. $\square$

2) Classification, application & generalization

2.1) Classification.

Suppose $X := \operatorname{Spec}(A)$ is a conical symplectic singularity.

We write $HC(\mathcal{A})$ for the full subcategory of $\operatorname{Bimod}(\mathcal{A})$ consisting of HC bimodules. For $\mathcal{B} \in HC(\mathcal{A})$, pick a good filtration on $\mathcal{B}$ and consider $\operatorname{Supp}(\operatorname{gr}\mathcal{B}) \subset X$ i.e. the subvariety defined by the annihilator of $\operatorname{gr}\mathcal{B}$.

Exercise: 1) Use Remark in Sec 1.1 to show that $\operatorname{Supp}(\operatorname{gr}\mathcal{B})$ is independent of the choice of a good filtration. We write $\operatorname{Supp}(\mathcal{B})$ for $\operatorname{Supp}(\operatorname{gr}\mathcal{B})$

2*) Let $\mathcal{I}$ be the left annihilator of $\mathcal{B}$. Then $\operatorname{Supp}(\mathcal{A}/\mathcal{I}) = \operatorname{Supp}(\mathcal{B})$.
Hint: $\mathcal{A}/\mathcal{I} \hookrightarrow \operatorname{End}_{\mathcal{A}^{opp}}(\mathcal{B})$ & a good filtr'n of $\mathcal{B}$ induces one on the target.

3) If $\mathcal{A}$ is simple, then $\operatorname{Supp}(\mathcal{B}) = X$ if $\mathcal{B} \neq 0$.

Definition: The category of HC bimodules with full support, denoted by $\overline{HC}(\mathcal{A})$, is the Serre quotient
$$\overline{HC}(\mathcal{A}) := HC(\mathcal{A}) / \{\mathcal{B} \mid \operatorname{Supp}(\mathcal{B}) \neq X\}$$

It turns out that $\overline{HC}(\mathcal{A})$ is equivalent to the category of representations of a finite group that is a quotient of the algebraic fundamental group $\pi_1^{alg}(X^{reg})$. It can be defined as the pro-finite completion of the usual fundamental group $\pi_1(X^{reg})$. It controls the finite etale covers of X^{reg} in the same way $\pi_1(X^{reg})$ controls topological covers.

Fact: $\pi_1^{alg}(X^{reg})$ is finite (Namikawa). Hence $\pi_1(X^{reg}) \twoheadrightarrow \pi_1^{alg}(X^{reg})$ and every finite dimensional representation of $\pi_1(X^{reg})$ factors through $\pi_1(X^{reg})$.

Examples: 1) Let $\widetilde{\mathbb{O}} = G/H$, where G is simply connected, be an equivariant cover of a nilpotent orbit, and $X = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$. Then $\pi_1(X^{reg}) = \pi_1(\widetilde{\mathbb{O}}) = H/H^\circ$, and $\pi_1^{alg}(X^{reg}) = \pi_1(X^{reg})$.

2) Let $\Gamma \subset Sp(V)$ be a finite subgroup & $X := V/\Gamma$. Then $\pi_1^{alg}(X^{reg}) = \pi_1(X^{reg}) = \Gamma$.

In the general case we'll write Γ for $\pi_1^{alg}(X^{reg})$.

Thm (I.L.18): There is a normal subgroup $\Gamma_{\mathcal{A}} \triangleleft \Gamma$ s.t.

$$\overline{HC}(\mathcal{A}) \xrightarrow{\sim} \operatorname{Rep}(\Gamma/\Gamma_{\mathcal{A}})$$

Moreover, under mild assumptions on X, one can recover $\Gamma_{\mathcal{A}}$ from the quantization parameter of $\mathcal{A}$, an element of $\mathfrak{h}_X/W_X$.

Example 2: Let $\mathcal{A}=\mathcal{A}_0$ be the canonical quantization. Let $\tilde{X}^\circ$ be cover of X^{reg} corresponding to $\Gamma_{\mathcal{A}_0} \trianglelefteq \Gamma$. Set $\tilde{X} := \operatorname{Spec} \mathbb{C}[\tilde{X}^\circ]$ so that $\Gamma/\Gamma_{\mathcal{A}_0} \curvearrowright \tilde{X}$ w. $X = \tilde{X}/(\Gamma/\Gamma_{\mathcal{A}_0})$. Let $\mathcal{L}_1, \ldots \mathcal{L}_k$ be all codim 2 leaves in X and set $X^1 := X^{reg} \sqcup \coprod_{i=1}^{k} \mathcal{L}_i$ so that $\operatorname{codim}_X X \setminus X^1 \geqslant 4$. Assume none of the slices Σ_i to $\mathcal{L}_i$ has type E_8. Then $\tilde{X}$ is the maximal cover of X unramified over X^1 ($\tilde{X} \to X$ factors through every other such cover), [LMMB'21].

Let Γ' be the Galois group of $\tilde{X} \to X$. We construct a functor $\operatorname{Rep}(\Gamma') \longrightarrow HC(\mathcal{A})$ that then gives rise to the desired equivalence $\operatorname{Rep}(\Gamma') \xrightarrow{\sim} \overline{HC}(\mathcal{A})$. The variety $\tilde{X}$ is a conical symplectic singularity, let $\tilde{\mathcal{A}}_0$ be its canonical quantization. The Γ'-action of $\mathbb{C}[\tilde{X}]$ extends to $\tilde{\mathcal{A}}_0$ & $\mathcal{A}_0 = \tilde{\mathcal{A}}_0^{\Gamma'}$. To a representation τ of Γ' we assign the HC $\mathcal{A}_0$-bimodule

$$\mathcal{B}_\tau := (\tau \otimes \tilde{\mathcal{A}}_0)^{\Gamma'}$$

2.2) Application.

The main application (at this point) of the classification thm is to defining and studying unipotent HC bimodules for $\mathcal{U}(\mathfrak{g})$ – those that should correspond to nilpotent orbits & their covers under the non-existing Orbit method.

Thm/definition (LMBM'21): Let $\tilde{\mathbb{O}}$ be an equivariant cover of a nilpotent orbit in $\mathfrak{g}^*$ & $\mathcal{A}_0$ is the canonical quantization of $\mathbb{C}[\tilde{\mathbb{O}}]$. Then $\mathcal{A}_0$ is semisimple as a $\mathcal{U}(\mathfrak{g})$-bimodule. The simple direct summands are called unipotent bimodules.

One can say a lot about unipotent bimodules using the general theory. One can describe the annihilators = kernels of $\varphi: U(\mathfrak{g}) \to \mathcal{A}_0$. By Thm in the end of Sec 1.2 of Lec 4, they are maximal ideals. One can compute their "infinitesimal characters," i.e the corresponding points of $\mathfrak{h}^*/W$. One can show that the unipotent bimodules w. given annihilator are classified by irreducible representations of a certain finite group – this is where the classification thm comes into play. Finally, one can show that most of unipotent bimodules are unitarizable.

2.3) Techniques of proof.

There are two key steps:

Step i): Produce a full embedding $\overline{HC}(\mathcal{A}) \hookrightarrow Rep(\Gamma)$. Both categories are monoidal & so is the functor. It will follow that the image is $Rep(\Gamma/\Gamma_{\mathcal{A}})$ for a unique $\Gamma_{\mathcal{A}} \triangleleft \Gamma$.

We will only explain what the functor does to objects. Pick $\mathcal{B} \in HC(\mathcal{A})$ and equip it w. a good filtration. The $gr\,\mathcal{B}$ is a finitely generated A-module. But it comes w. an additional structure, a Poisson bracket map $\{\cdot,\cdot\}: A \times B \to B$ given by

$$\{a+\mathcal{A}_{\leq i-1}, b+\mathcal{B}_{\leq j-1}\} := [a,b] + \mathcal{B}_{\leq i+j-d-1}.$$

It satisfies usual axioms. So $gr\,\mathcal{B}$ becomes a "Poisson A-module." We view $gr\,\mathcal{B}$ as a coherent sheaf on X. Consider the restriction $gr\,\mathcal{B}|_{X^{reg}}$. Recall that X^{reg} is a (smooth) symplectic variety.

Fact: Let Z be a smooth symplectic variety & M be a Poisson $\mathcal{O}_Z$-module. Then M has the unique $\mathcal{D}_Z$-module structure s.t. for a local function f, the vector field $\{f,\cdot\}$ on Z acts as the bracket w. f w.r.t. the Poisson module structure.

So $(\operatorname{gr}\mathcal{B})|_{X^{reg}}$ becomes an $\mathcal{O}$-coherent $\mathcal{D}$-module, i.e. a vector bundle w. a flat connection. Pick $x\in X^{reg}$. The fiber $(\operatorname{gr}\mathcal{B})_x$ carries a representation of $\pi_1(X^{reg},x)$ that must factor through $\pi_1^{alg}(X^{reg})$. On objects, our functor is $\mathcal{B}\mapsto(\operatorname{gr}\mathcal{B})_x$.

It's not clear from this construction why it's a functor. The actual construction of the full embedding is more involved.

Step ii: We need to figure out when $V\in \operatorname{Rep}\Gamma$ lies in the image. This can be reinterpreted as follows. We can "microlocalize" $\mathcal{A}$ to X^{reg} getting, roughly speaking, a sheaf of filtered algebras on X^{reg}, to be denoted by $\mathcal{A}^{reg}$. We can still talk about HC bimodules over $\mathcal{A}^{reg}$. Moreover, $V\in\operatorname{Rep}\Gamma$ gives rise to an $\mathcal{O}$-coherent $\mathcal{D}$-module on X^{reg}, hence to a Poisson $\mathcal{O}_{X^{reg}}$-module. One can uniquely quantize it to a sheaf of $\mathcal{A}^{reg}$-bimodules on X^{reg} to be denoted by $\mathcal{B}_V^{reg}$.

Our question is when $\mathcal{B}_V^{reg}$ comes as the restriction (="microlocalization") of a HC $\mathcal{A}$-bimodule. In other words, we need to know when $\mathcal{B}_V^{reg}$ extends to a HC $\mathcal{A}$-bimodule.

A classical fact about coherent sheaves in Algebraic geometry is that one can always extend such a sheaf from an open subvariety to an ambient variety. Sadly, this is not the case for coherent modules over quantizations: it may happen that $\Gamma(\mathcal{B}_V^{reg})=\{0\}$.

To study the question of when one can extend $\mathcal{B}_V^{reg}$ to X we'll need constructions from Sec 3.2 of Lec 3 recalled in Example 2 in Sec 2.1: $X^1 = X^{reg} \sqcup \bigsqcup_{i=1}^{k} \mathcal{L}_i$.

Let $\iota: X^{reg} \hookrightarrow X^1$ be the inclusion. We need to see for which V the following holds

(*) $\mathcal{B}_V^{reg}$ extends "nicely" to X^1, i.e. that the sheaf-theoretic pushforward $\iota_* \mathcal{B}_V^{reg}$ is coherent in a suitable sense.

Once this is known, set $\mathcal{B}_V = \Gamma(\iota_* \mathcal{B}_V^{reg})$. Then we have $\mathcal{B}_V|_{X^{reg}} \simeq \mathcal{B}_V^{reg}$.

To address (*) we show that the question of when $\iota_* \mathcal{B}_V^{reg}$ is coherent around $\mathcal{L}_i$ can be reduced to a slice Σ_i to $\mathcal{L}_i$, the neighborhood of 0 in $\mathbb{C}^2/\Gamma_i$. In more detail, let $\lambda \in \mathfrak{h}_X$ be the parameter of $\mathcal{A}$. By Sec 3.2 of Lec 3, $\mathfrak{h}_X = \bigoplus_{j=0}^{k} \mathfrak{h}_j$ w. $\mathfrak{h}_j \hookrightarrow \mathfrak{h}_{\Gamma_j} = \mathfrak{h}_{\mathbb{C}^2/\Gamma_j}$ $\forall j=1,\dots k$. Let λ_i be the $\mathfrak{h}_i$-component of λ, and $\mathcal{A}_{i,\lambda_i}$ the corresponding quantization of $\mathbb{C}^2/\Gamma_i$.

Note that the inclusion $\Sigma_i \hookrightarrow X$ gives rise to the group homomorphism $\Gamma_i = \pi_1^{alg}(\Sigma_i \setminus \{0\}) \xrightarrow{\varphi_i} \pi_1^{alg}(X^{reg}) = \Gamma$. Then $\mathcal{B}_V^{reg}$ is coherent around $\mathcal{L}_i$ iff there's a HC $\mathcal{A}_{i,\lambda_i}$-bimodule mapping to $\varphi_i^*(V)$ under our functor. One can analyze the latter as long as Γ_i is not of type E_8 (I.L. 2018).

2.4) Generalization.

One could try to generalize the classification thm in Sec 2.1 to the problem of "quantizing singular Lagrangians." Namely, let X be as before. Consider a subvariety $Y \subset X$ satisfying the following:

(i) $\forall$ symplectic leaf $\mathcal{L} \subset X \Rightarrow Y \cap \mathcal{L} \subset \mathcal{L}$ is isotropic (i.e the restriction of the symplectic form from $\mathcal{L}$ to $(Y \cap \mathcal{L})^{reg}$ is zero).

(ii) $\overline{Y \cap X^{reg}} = Y$.

(iii) Y is stable under the contracting $\mathbb{C}^\times$-action.

We often impose an additional condition:

(iv) Y is irreducible & $\operatorname{codim}_Y Y^{sing} \geqslant 2$.

Example: Consider the Poisson variety $X \times X^{opp}$, where "opp" means that we multiply $\{\cdot,\cdot\}$ by -1. Then $X_{diag} \subset X \times X^{opp}$ satisfies (i)-(iv).

One could ask to quantize Y (equipped with an additional structure) to a module over a fixed quantization $\mathcal{A}$.

If (iv) holds, then for the additional structure we can take a vector bundle w. a twisted flat connection on Y^{reg} (e.g. in the example there's no twist). We are looking for $\mathcal{A}$-modules $\mathcal{M}$ s.t. $\forall$ "good filtration" on $\mathcal{M}$, the restriction $\operatorname{gr}\mathcal{M}|_{Y^{reg}}$ is the fixed vector bundle w. twisted flat connection. Such $\mathcal{M}$ is expected to be unique if it exists and the question is when it exists. This is expected to be reduced to the case when $\dim X = 4$ (and hence $\dim Y = 2$).

Using this ideology, I.L. & S. Yu have classified HC modules over quantizations of $\mathbb{C}[\mathbb{O}]$, where $\operatorname{codim}_{\mathbb{O}} \partial\mathbb{O} \geqslant 4$. If we remove this condition, then (iv) no longer holds and the situation becomes complicated...