EXERCISES FOR LECTURE 1

Section 1

Below in this section G is a Lie group and $\mathfrak{g}$ is its Lie algebra.

Exercise 1. Show that any classical comoment map is a Lie algebra homomorphism.

Exercise 2. The purpose of this exercise is to establish a symplectic structure on a coadjoint G-orbit $G\alpha \subset \mathfrak{g}^*$ and to show that the G-action is Hamiltonian.

Note that if M is a Poisson manifold, then the Poisson bracket on $C^\infty(M)$ can be viewed as a bivector field, i.e., a section of $\Lambda^2 T_M$. We will denote it by $\mathcal{P}$.

1) Show that $\mathcal{P}_\alpha$ is contained in the subspace $\Lambda^2 T_\alpha G\alpha$ of $\Lambda^2 T_\alpha \mathfrak{g}^*$ and is a nondegenerate element in that subspace. Show that there is a unique G-invariant bivector field on $G\alpha$ whose fiber at α is $\mathcal{P}_\alpha$. Moreover, check that this bivector field comes from a symplectic form on $G\alpha$. This equips $G\alpha$ with a symplectic structure so that G acts by symplectomorphisms.

2) Show that the resulting symplectic form ω on $G\alpha$ satisfies $\omega_\alpha(\xi.\alpha, \eta.\alpha) = \langle \alpha, [\xi, \eta]\rangle$, for all $\xi, \eta \in \mathfrak{g}$ and $\xi.\alpha$ means the image of α under ξ.

3) Show that the inclusion $G\alpha \hookrightarrow \mathfrak{g}^*$ is a moment map for the G-action.

Exercise 3. Let M be a Poisson manifold with a transitive Hamiltonian G-action. Let $\mu : M \to \mathfrak{g}^*$ be the moment map. Prove that

1) $\operatorname{im}\mu \subset \mathfrak{g}^*$ is a single orbit.

2) $\mu : M \to \operatorname{im}\mu$ is a cover and $\mu^* : C^\infty(\operatorname{im}\mu) \to C^\infty(M)$ intertwines the Poisson brackets.

3) The Poisson structure on M is nondegenerate, and μ is a symplectomorphism.

Section 2

Exercise 1. Let $\mathcal{A} = \bigcup_{i\geqslant 0} \mathcal{A}_{\leqslant i}$ be a filtered algebra with $\deg[\cdot,\cdot] \leqslant -d$, i.e, $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-d}$ for all i, j. Show that the bracket on the associated graded algebra $\operatorname{gr}\mathcal{A}$ given on the homogeneous elements $a + \mathcal{A}_{\leqslant i-1}, b + \mathcal{A}_{\leqslant j-1}$ (with $a \in \mathcal{A}_{\leqslant i}, b \in \mathcal{A}_{\leqslant j}$) by

$$\{a + \mathcal{A}_{\leqslant i-1}, b + \mathcal{A}_{\leqslant j-1}\} \subset [a, b] + \mathcal{A}_{\leqslant i+j-d-1}$$

is a Poisson bracket.

Exercise 2. Let V be a (finite dimensional) symplectic vector space with form ω and $W(V)$ be its Weyl algebra,

$$W(V) := T(V)/(u \otimes v - v \otimes u - \omega(u, v)).$$

Prove that $W(V)$ is the unique filtered quantization of the graded Poisson algebra $S(V)$ (with $d = 2$).

Section 3

In this section G is a semisimple algebraic group (over $\mathbb{C}$) and $\mathfrak{g}$ is its Lie algebra.

Exercise 1. Let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Show that e and f are nilpotent.

Exercise 2. This exercise deals with the classification of nilpotent orbits in the classical Lie algebras of types B,C,D under the full orthogonal/ symplectic group.

1) Show that a finite dimensional representation of $\mathfrak{sl}_2$ has an invariant orthogonal (resp., symplectic) form iff every even (resp., odd) dimensional irreducible representation occurs with even multiplicity. *Hints: first show that a representation of this form has an invariant form of the specified type. Then show that if $U_1 \oplus U_2$ and U_1 both have an invariant, say, orthogonal form, then so does U_2 (even if the form on U_1 is not the restriction of the form on $U_1 \oplus U_2$).*

2) Show that an invariant orthogonal (or symplectic) form on a finite dimensional representation of $\mathfrak{sl}_2$ is unique up to an $\mathfrak{sl}_2$-linear isomorphism.

3) Conclude that the nilpotent O_n- (resp., Sp_{2n}-) orbits in $\mathfrak{so}_n$ (resp., $\mathfrak{sp}_n$) are classified by the partitions of n, where every even (resp., odd) part occurs with even multiplicity.

Exercise 3. See also Section 1.2 of Lecture 2. For the purposes of classifying orbits of SO_n (as opposed to O_n) and many others we need to understand the centralizers $Z_G(e)$, where $G = \mathrm{O}_n$ or Sp_n. Until the further notice G is a general reductive algebraic group.

Fix an $\mathfrak{sl}_2$-triple (e, h, f). Let

$$\mathfrak{g}_i := \{x \in \mathfrak{g} | [h, x] = ix\}, \mathfrak{z}_\mathfrak{g}(e)_i := \mathfrak{z}_\mathfrak{g}(e) \cap \mathfrak{g}_i, \mathfrak{z}_\mathfrak{g}(e)_{>0} := \bigoplus_{i>0} \mathfrak{z}_\mathfrak{g}(e)_i.$$

1) Show that $\mathfrak{z}_\mathfrak{g}(e) = \mathfrak{z}_\mathfrak{g}(e, h, f) \oplus \mathfrak{z}_\mathfrak{g}(e)_{>0}$, that $\mathfrak{z}_\mathfrak{g}(e)_{>0}$ is the Lie algebra of a unipotent subgroup of G, to be denoted by $Z_G(e)_{>0}$ and that, finally, $Z_G(e) = Z_G(e, h, f) \ltimes Z_G(e)_{>0}$.

2) Suppose now that $G = \mathrm{O}_n$ or Sp_n. Let e be a nilpotent element, and $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ be the corresponding partition. Show that $Z_G(e, h, f)$ is the group of orthogonal/ symplectic automorphisms of the corresponding representation of $\mathfrak{sl}_2$ and use this to identify $Z_G(e, h, f)$ with $\prod_{i=1}^n G_i$, where G_i is as follows:

- For $G = \mathrm{O}_n$, the group G_i is O_{m_i} is i is odd and Sp_{m_i} if i is even.
- For $G = \mathrm{Sp}_n$, the group G_i is O_{m_i} is i is even and Sp_{m_i} if i is odd.

Exercise 4. Use Exercise 3 to show that the following conditions are equivalent:

- A nilpotent O_n-orbit in $\mathfrak{so}_n$ splits into the disjoint union of two distinct SO_n-orbits.
- $Z_{\mathrm{O}_n}(e, h, f) \subset \mathrm{SO}_n$.
- All parts in the corresponding partition are even.