EXERCISES FOR LECTURE 2

Lecture 1 overflow

Below in this section G is a complex semisimple algebraic group, and $\mathfrak{g}$ is its Lie algebra.

Exercise 1. Show that the nilpotent cone (= the subset of all nilpotent elements) is Zariski closed in $\mathfrak{g}$.

Exercise 2. Show that every nilpotent orbit in $\mathfrak{g}$ is stable under the dilation $\mathbb{C}^\times$-action.

Exercise 3. Use 2) of the 2nd exercise for Sec 1 in Lec 1, to show that the degree of the Poisson bracket on $\mathbb{C}[\mathbb{O}]$ is -1.

Section 1

Exercise 1. The exercise concerns the example of the action of $G = \mathrm{Sp}_{2n}$ on $\mathbb{C}^{2n}$.

1) Show that $\mu : \mathbb{C}^{2n} \to \mathfrak{g}^*$ given by $\langle \mu(v), \xi\rangle = \frac{1}{2}\omega(\xi v, v)$ is a moment map.
2) Show that $\operatorname{im}\mu$ is the closure of the orbit $\mathbb{O}$ corresponding to the partition $(2, 1^{2n-2})$.
3) Show that over $\mathbb{O}$, the morphism μ is a 2-fold cover.

Section 2.3

Exercise 1. Let $P \subset G$ be a parabolic subgroup. Show that the morphism $\mu : T^*(G/P) \to \mathfrak{g}^*$ given by $[g, \alpha] \mapsto g.\alpha$ (the image of α under the action of g) is proper.

Exercise 2. Let $\mathbb{O}$ be a nilpotent orbit in $\mathfrak{g}$. The purpose of this exercise is to show that $X := \mathrm{Spec}(\mathbb{C}[\mathbb{O}])$ has symplectic singularities. For this we will explicitly construct a resolution Y as in the definition in Section 2.1 of Lecture 2. Recall that we write $\mathfrak{g}_i$ for $\{x \in \mathfrak{g} | [h, x] = ix\}$. For $j \in \mathbb{Z}$ we write $\mathfrak{g}_{\geqslant j} := \bigoplus_{i \geqslant j} \mathfrak{g}_i$ and define $\mathfrak{g}_{\leqslant j}$ similarly. Let P denote the connected subgroup with Lie algebra $\mathfrak{g}_{\geqslant 0}$, note that it's parabolic. Define Y as the homogeneous vector bundle $G \times^P \mathfrak{g}_{\geqslant 2}$, where the points are equivalence classes $[g, x]$ with $[g, x] = [gp^{-1}, \mathrm{Ad}(p)x]$.

1) Consider the morphism $\pi : Y \to \mathfrak{g}, [g, x] = \mathrm{Ad}(g)x$. Show that it's proper, its image is $\overline{\mathbb{O}}$, and π is an isomorphism over $\mathbb{O}$.
2) Note that $T_{[1,x]}Y$ is naturally identified with $\mathfrak{g}_{\leqslant -1} \oplus \mathfrak{g}_{\geqslant 2}$, where $\mathfrak{g}_{\geqslant 2}$ is embedded as the subspace of vectors tangent to the fiber of $Y \to G/P$, while $\mathfrak{g}_{\leqslant -1}$ is embedded via the action map, $\xi \mapsto \xi_Y$. Show that, for $y \in \mathfrak{g}_{\leqslant -1}, z \in \mathfrak{g}_{\geqslant 2}$, we have $d_{[1,x]}(y, z) = [y, x] + z$.
3) Show that there is a unique G-invariant 2-form $\tilde{\omega}$ on Y such that for $x \in \mathfrak{g}_{\geqslant 2}, y_1, y_2 \in \mathfrak{g}_{\leqslant -1}, z_1, z_2 \in \mathfrak{g}_{\geqslant 2}$ we have

$$\tilde{\omega}_{[1,x]}(y_1 + z_1, y_2 + z_2) = (x, [y_1, y_2]) + (y_1, z_2) - (y_2, z_1),$$

where we write $(\cdot, \cdot)$ for the Killing form. Moreover, show that $\tilde{\omega}$ extends $\pi^*\omega$, where ω is the Kirillov-Kostant form on Ge.
4) Conclude that X is singular symplectic.

Exercise 3. Let P be a parabolic subgroup of G. Let $\widetilde{\mathbb{O}}$ be the open G-orbit in $T^*(G/P)$ and $X := \operatorname{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$. Show that $T^*(G/P)$ is a symplectic resolution of X (hint: use the Stein factorization for $\mu : T^*(G/P) \to \mathfrak{g}^*$).

Exercise 4. Consider the group $G = \mathrm{Sp}_4$ and the nilpotent orbit $\mathbb{O}$ corresponding to the partition $(2, 2)$. Further, let P_1 be the parabolic subgroup of G stabilizing a line in $\mathbb{C}^4$, and P_2 be the parabolic subgroup stabilizing a lagrangian (=2-dimensional isotropic) subspace in $\mathbb{C}^4$. Let $\tilde{\mathbb{O}}_i$ denote the open orbit in $T^*(G/P_i)$. Show that $\tilde{\mathbb{O}}_1$ is a 2-fold cover of $\mathbb{O}$, and $\tilde{\mathbb{O}}_2$ is $\mathbb{O}$.

Section 3

The goal of the only exercise in this section is to elaborate on the isomorphism $\operatorname{gr}\mathcal{U}_\lambda \cong \mathbb{C}[\mathcal{N}]$. We will need the following facts:

- We have a graded algebra isomorphism $\mathbb{C}[\mathfrak{g}^*]^G \cong \mathbb{C}[\mathfrak{h}^*]^W$ – this is the Chevalley restriction theorem. In particular, there are free homogeneous generators $f_1, \ldots, f_r \in \mathbb{C}[\mathfrak{g}]^G$, where $r = \dim \mathfrak{h}$.
- The subvariety $\mathcal{N} \in \mathfrak{g}^*$ is normal, has codimension r, and its ideal of zeroes is $(f_1, \ldots, f_r)$. These results are due to Kostant. In particular, $f_1, \ldots, f_r$ form a *regular sequence* in $\mathbb{C}[\mathfrak{g}^*]$.
- Since $f_1, \ldots, f_r$ form a regular sequence, the first homology of the associated Koszul complex vanishes. Explicitly, this means that if $g_1, \ldots, g_r \in \mathbb{C}[\mathfrak{g}^*]$ satisfy $\sum_{i=1}^r f_i g_i = 0$, then there are elements $g_{ij} \in \mathbb{C}[\mathfrak{g}^*]$ satisfying $g_{ij} = -g_{ji}$ and $g_i = \sum_{j=1}^r g_{ij} f_j$.

Exercise. Establish a graded Poisson algebra epimorphism $\mathbb{C}[\mathcal{N}] \twoheadrightarrow \operatorname{gr}\mathcal{U}_\lambda$ and show it is an isomorphism.