EXERCISES FOR LECTURE 3

Section 1

Exercise. Suppose X is a singular symplectic variety, while Y is a normal Poisson variety equipped with a proper birational Poisson morphism to X. Prove that Y is also singular symplectic. *Hint: you need the theorem that a resolution of singularities (resolving exactly the singular locus) exists. Also this exercise is more Algebro-geometric than many exercise for this mini-course.*

Section 2

We use the notation of Section 2 in Lecture 3.

Exercise 1. *This is a technical exercise.* Consider the subvariety

$$\mu^{-1}(0) = \{(g, \alpha, x) \in G \times \mathfrak{g}^* \times X_L | \alpha|_{\mathfrak{p}} = \mu_L(x)\}$$

It is acted on by P. On the other hand, consider the variety $X_L \times (\mathfrak{g}/\mathfrak{p})^*$ and equip with an action of $P = L \ltimes U$ as follows: the group L acts diagonally, while U acts by $u.(x, \beta) = (x, u.\beta + u\mu_L(x) - \mu_L(x))$. Verify that this indeed an action of P and establish a P-equivariant isomorphism

$$\{(\alpha, x) \in \mathfrak{g}^* \times X_L | \alpha|_{\mathfrak{p}} = \mu_L(x)\} \xrightarrow{\sim} X_L \times (\mathfrak{g}/\mathfrak{p})^*$$

Exercise 2. *This is the construction of classical Hamiltonian reduction.* Let A be a Poisson algebra equipped with a rational action of an algebraic group G by Poisson algebra automorphisms. Assume, further, that there is a classical comoment map $\varphi : \mathfrak{g} \to A$ for this action. Show that there is a unique Poisson bracket on $(A/A\varphi(\mathfrak{g}))^G$ satisfying

$$\{a + A\varphi(\mathfrak{g}), b + A\varphi(\mathfrak{g})\} = \{a, b\} + A\varphi(\mathfrak{g}).$$

Exercise 3. *Another technical exercise.* Let $\omega : G \to G/P, \eta : Y = \mu^{-1}(0)/P \to G/P$ be the natural projections. Show that, for an open affine subset $U \subset G/P$, we have a natural identification of $\mathbb{C}[\eta^{-1}(U)]$ with the Hamiltonian reduction for the action of P on $\mathbb{C}[T^*\omega^{-1}(U) \times X_L]$ (with moment map μ).

Exercise 4. Let $\mu' : Y \to \mathfrak{g}^*$ be the morphism defined in Section 2. Show that

(1) μ' is a moment map for the G-action on Y.
(2) μ' is proper.
(3) $\operatorname{im} \mu'$ is the closure of a single nilpotent orbit.

Exercise 5. Show that if X_L is $\mathbb{Q}$-factorial terminal, then $Y = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*)$ is as well.