EXERCISES FOR LECTURE 4

Section 1.1

Exercise 1. Let L be a connected reductive group, and $\widetilde{\mathbb{O}}_L$ be an L-equivariant cover of a nilpotent orbit in $\mathfrak{l}^*$. Suppose that the degree of the Poisson bracket on $\mathbb{C}[\widetilde{\mathbb{O}}_L]$ is $-d$ so that we have the classical comoment map $\varphi : \mathfrak{l} \to \mathbb{C}[\widetilde{\mathbb{O}}_L]_d$. Finally, let $\mathcal{A}$ be a filtered quantization of $\mathbb{C}[\widetilde{\mathbb{O}}_L]$.

1) Show that there is a Lie algebra homomorphism $\Phi : \mathfrak{l} \to \mathcal{A}_{\leqslant d}$ such that $\varphi = \Phi + \mathcal{A}_{\leqslant d-1}$. Moreover, show that the additional condition that Φ vanishes on the center $\mathfrak{z}(\mathfrak{l})$ specifies Φ uniquely.

2) Show that the action of L on $\mathbb{C}[\widetilde{\mathbb{O}}_L]$ lifts to a rational G-action on $\mathcal{A}$ by filtered algebra automorphisms such that Φ is a quantum comoment map for this action.

Exercise 2. Let $\mathcal{A}$ be an associative algebra equipped with a rational action of an algebraic group R by algebra homomorphisms. Let Φ be a quantum comoment map for this action. Show that the space $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{r}))^R$ carries a unique associative product given by $(a + \mathcal{A}\Phi(\mathfrak{r}))(b + \mathcal{A}\Phi(\mathfrak{r})) = ab + \mathcal{A}\Phi(\mathfrak{r})$. Here a, b are elements of $\mathcal{A}$ such that $a + \mathcal{A}\Phi(\mathfrak{r}), b + \mathcal{A}\Phi(\mathfrak{r})$ are R-invariant.

Section 2

Here $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$.

Exercise. Let V be a finite dimensional representation of $\mathfrak{g}$. Show that

1) $V \otimes U(\mathfrak{g})$ has a unique $\mathfrak{g}$-bimodule structure given by $(v \otimes a)x := v \otimes (ax)$ and $x(v \otimes a) = (x.v) \otimes a + v \otimes xa$. Show that this defines a HC $U(\mathfrak{g})$-bimodule structure on $V \otimes U(\mathfrak{g})$.

2) Moreover, show that any HC $U(\mathfrak{g})$-bimodule is isomorphic to a quotient of a bimodule of the form described in 1).