Lecture 1: Intro to representations of groups.

1) Concept of a group representation.
2) Irreducible representations.
3) Characters.
4) Applications to the structure of finite groups.
5) Example: from actions to representations.

1) Concept of a representation.

In the study of general groups some groups play a distinguished role. The most basic example: to a set X we assign the group $Bij(X) = \{\text{bijections } X \to X\}$ w.r.t. composition. If X is a finite set, say $\{1,2,\dots n\}$, then we recover the symmetric group S_n.

One way how $Bij(X)$ is distinguished is that, for a group G, we want to "compare" G to $Bij(X)$ by studying group homomorphisms $G \to Bij(X)$. Note that to give such a homomorphism, say ρ, is the same thing is to define an action of G on X: the action map $G \to X$ is given by

$(g,x)\mapsto \rho(g)(x)$. Conversely, given an action $(g,x)\mapsto g.x$, we observe that the map $\rho_g : x \mapsto g.x$ is a bijection and the resulting assignment $g\mapsto \rho_g : G \longrightarrow \mathrm{Bij}(X)$ is a group homomorphism. This establishes a 1-1 correspondence between G-actions on X & group homomorphisms $G \longrightarrow \mathrm{Bij}(X)$. Details are an exercise.

The importance of group actions for understanding the structure of groups is already seen, e.g., in the proof of Sylow's theorems. This justifies the claim that it's a good idea to consider homomorphisms $G\longrightarrow \mathrm{Bij}(X)$.

When X comes with an additional structure, it makes sense to restrict to bijections preserving this structure. In this course we will be interested in a special case of this situation. Let $\mathbb{F}$ be a field. We will mostly use $\mathbb{F}=\mathbb{C}$. But we can also take $\mathbb{F}=\mathbb{R}$ or one of positive characteristic fields such as $\mathbb{F}=\mathbb{Z}/p\mathbb{Z}$. Let X carry the structure of a vector space over $\mathbb{F}$. Then the relevant subgroup of $\mathrm{Bij}(X)$ is the group of all invertible linear operators $X\to X$ usually denoted by $GL(X)$ (and called the general linear group).

If $\dim_{\mathbb{F}} X = n$ and we fix an isomorphism $X \xrightarrow{\sim} \mathbb{F}^n$, then $GL(X)$ is identified w. the group $GL_n(\mathbb{F})$ of invertible matrices of size n.

Definition: A representation of a group G in an $\mathbb{F}$-vector space X is a group homomorphism $G \to GL(X)$.

Similarly to the correspondence between homomorphisms to $Bij(X)$ and actions on X, a representation can be thought of as an action on X by linear operators. The goal of this course, not surprisingly, is to study representations.

2) Irreducible representations.

A basic structural fact about an action of a group G on a set X is that X decomposes as the disjoint union of orbits (and an orbit, of course, cannot be decomposed further). We would like to find some analog of this for representations. The role of smallest building blocks (such as orbits for actions) in this world is played by so called "irreducible representations."

Definition: A representation of G in V is called *irreducible* if the only two G-stable subspaces are $\{0\} \neq V$.

The most basic analog of taking unions in the world of representations is taking direct sums: given representations of G in vector spaces $V_1, \dots V_k$, we get a natural representation in $V_1 \oplus V_2 \oplus \dots \oplus V_k$, in terms of linear actions it's given by

$$g.(v_1, \dots v_k) := (g.v_1, \dots, g.v_k).$$

The following result is of fundamental importance.

Thm (Maschke) Suppose

1) $\mathrm{char}\,\mathbb{F} = 0$ (e.g. $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$)
2) and G is a finite group.

Then any finite dimensional representation of G decomposes into the direct sum of irreducible representations.

We'll prove this later – and we'll also see that both conditions 1) & 2) are necessary.

3) Characters.

Suppose the assumptions of the theorem hold. Then we can ask the following questions.

Q1: Can we classify (finite dimensional) irreducible representations (i.e. find a set, possibly of combinatorial nature, with a bijection to the set of irreducibles)?

Q2: For a given irreducible reprentation, V, compute its numerical invariants, such as dimension.

Q3: For an arbitrary (finite dimensional) representation find its decomposition into the direct sum of irreducibles.

As it turns out all these questions are addressed using the same ingredient – "characters."

Definition: Let ρ be a representation of G in a finite dimensional space V. Then its character $\chi_\rho\,(=\chi_V)$ is the function on G defined by $\chi_V(g) = tr\ \rho(g)$, where, recall tr (trace) is the sum of diagonal entries of the matrix of $\rho(g)$ (in any basis).

A connection to Q2 is quite straightforward: $\dim V = \operatorname{tr} 1_V = \chi_V(1)$. We postpone explaining a connection to Q3, but sketch a connection to Q1. Note that, for $g, h \in G$,

$$\chi_V(hgh^{-1}) = [\rho \text{ is homomorphism}] = \operatorname{tr} \rho(h)\rho(g)\rho(h)^{-1} = [\operatorname{tr} \text{ is const. on conj. classes}] = \operatorname{tr} \rho(g) = \chi_V(g).$$

So, χ_V is constant of conjugacy classes in G. The following will be strengthened (and proved later).

Fact: Recall that G is finite & char $\mathbb{F} = 0$. Assume, in addition, that $\mathbb{F}$ is algebraically closed (e.g. $\mathbb{F} = \mathbb{C}$). The characters of irreducible representations form a basis in the space $Cl(G)$ of "class functions on G" i.e functions $G \to \mathbb{C}$ constant on conjugacy classes.

Note that $\dim Cl(G) = \#\{\text{conj. classes in } G\}$. This gives a count of the irreducible representations but not their classification: in general, there's no natural bijection between irreducible representations and conjugacy classes. In some cases, however, there is. Most notably, we will see that for $G=S_n$ both sets are naturally identified with the set of "partitions of n", a classical object in Combinatorics.

4) Applications to the structure of finite groups.

At this point one could – and should – ask a question about applications. One could expect, for example, applications to the structure theory of finite groups – just as for the actions. This is indeed the case.

A lot of attention in the study of finite groups was given to understanding "simple groups" – non-cyclic groups G w. exactly two different normal subgroups: $\{e\}$ & G. On the other hand, Sylow's theorems suggest that the structure of G is to some extent controlled by the decomposition of the order $|G|$ into primes. We will prove the following result later in the course.

Thm (Burnside): Let p,q be primes, and $a,b \in \mathbb{Z}_{\geq 0}$. A group of order $p^a q^b$ is not simple.

5) Example: from actions to representations.

Here we are going to give an example of a family of representations coming from actions on sets.

Let $\mathbb{F}$ be a field & X be a set. Consider the set $\mathrm{Fun}(X,\mathbb{F})$ of all functions $X\to\mathbb{F}$. It carries a natural vector space structure w. pointwise operations, e.g.

$$[f_1+f_2](x):=f_1(x)+f_2(x),\quad f_1,f_2\in \mathrm{Fun}(X,\mathbb{F}),\ x\in X.$$

Now let G be a group acting on X. We are going to equip $\mathrm{Fun}(X,\mathbb{F})$ with a representation of G by:

$$[g.f](x):=f(g^{-1}x)$$

Let's explain what goes into checking that this is a representation. We need to show that G acts on $\mathrm{Fun}(X,\mathbb{F})$ by linear operators:

$$g.(f_1+f_2)=g.f_1+g.f_2,\quad g.(af_1)=a(g.f_1),\ g\in G,\ f_1,f_2\in \mathrm{Fun}(X,\mathbb{F}),\ a\in\mathbb{F}.$$

This is done by evaluating both sides at $x\in X$ & is left as an exercise. Then we need to check the associativity $-\ g_1.(g_2.f)=(g_1g_2).f\ -$ and the unit $-\ e.f=f\ -$ axioms. We check the former, again by comparing the values at $x\in X$.

$$[g_1 . (g_2 . f)](x) = [g_2 . f](g_1^{-1}x) = f(g_2^{-1} g_1^{-1} x)$$

equal

$$[(g_1 g_2) . f](x) = f((g_1 g_2)^{-1} x) = f(g_2^{-1} g_1^{-1} x)$$

Our conclusion is that we get a representation of G in $\mathrm{Fun}(X, \mathbb{F})$. We'll talk more about this example in the next lecture.