Lecture 10: Characters, pt 3

1) Discussion of the theorem

2) Example

3*) More orthogonality!

1) Discussion of the theorem

Last time we have proved the following theorem

Theorem: Let $\mathbb{F}$ be algebraically closed field of char 0 & G be a finite group. Then the characters of irreducible representations of G form an orthonormal basis in

$$Cl(G)(=\{f\in Fun(X,\mathbb{F})\mid f(ghg^{-1})=f(h),\ \forall g,h\in G\})$$

w.r.t. the form $(f_1,f_2)=\frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g^{-1})$.

The proof was based on two claims, the latter of independent interest.

Claim 1: The characters of irreducibles span $Cl(G)$.

Claim 2: For representations U, V of G we have (in $\mathbb{F}$)

$$\dim \mathrm{Hom}_G(U,V) = (\chi_V, \chi_U).$$

In this section we will discuss how the theorem fails if $\mathbb{F}$ is not algebraically closed or has positive characteristic. We will also discuss a closely related result over $\mathbb{C}$.

1.1) Non-closed field.

Suppose $\mathbb{F}$ is not algebraically closed of char 0. The representations are completely reducible. From here with some work (including "base change to the algebraic closure") we can deduce Claim 2. Claim 1, however, may fail. For example, consider $G = \mathbb{Z}/3\mathbb{Z}$. In the setting, where Thm is true, there are 3 representations. The formula

$$|G| = \sum_{i=1}^{k} (\dim U_i)^2 / \dim \mathrm{End}_A(U_i)$$

((2) from Sec 2.1 in Lec 7, where $U_1, \dots, U_k$ are all irreducible representations of G) tells us then that $\dim U_i = 1$

for $i=1,2,3$. As discussed in Sec 1.1 of Lec 5, the 1-dimensional representations of $\mathbb{Z}/m\mathbb{Z}$ $(m=3)$ are in bijection w. cube roots of 1. And for $\mathbb{F}=\mathbb{R}$ there's just one. So Claim 1 is false.

Exercise: $\mathbb{Z}/3\mathbb{Z}$ has two irreducible representations over $\mathbb{R}$.

1.2) Positive characteristic fields.

Let $\operatorname{char}\mathbb{F}=p$. Claim 1 may fail. If $p \mid |G|$, then the number of irreducible representations is always less than the number of conjugacy classes. For example, if G is a p-group there's just one irreducible representation (Problem 3 in HW2). Claim 1 is still true when $\operatorname{char}\mathbb{F} \nmid |G|$.

And Claim 2 doesn't even make sense unless $\operatorname{char}\mathbb{F} \nmid |G|$ (b/c $(\cdot,\cdot)$ is undefined).

1.3) Field $\mathbb{C}$

For $\mathbb{F}=\mathbb{C}$, one can consider a Hermitian scalar product on $Cl(G)$ instead of a bilinear form:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)}$$

Thanks to the following lemma, we can use $\langle \cdot,\cdot \rangle$ instead of $(\cdot,\cdot)$ in the theorem.

Lemma: We have $\overline{\chi_V(g)} = \chi_V(g^{-1})$ for any G-representation V.

Proof: Let $\lambda_1, \dots \lambda_n$ be eigenvalues of g_V (w. multiplicities) so that $\chi_V(g) = \sum_{i=1}^n \lambda_i$, $\overline{\chi_V(g)} = \sum_{i=1}^n \overline{\lambda_i}$, $\chi_V(g^{-1}) = \sum_{i=1}^n \lambda_i^{-1}$

Note that $g^m = e \Rightarrow g_V^m = Id_V$ for some m. In particular, $\lambda_i^m = 1 \Rightarrow \overline{\lambda_i} = \lambda_i^{-1} \Rightarrow \overline{\chi_V(g)} = \chi_V(g^{-1})$ $\square$

2) Examples.

We record the following easy corollary of the theorem:

Corollary: Under the assumptions of the theorem, the number of irreducible representations of G (up to isomorphism) is the number of conjugacy classes in G.

Rem: This is the coincidence of numbers. There's no natural bijections between the sets. For example, consider $G=\mathbb{Z}/m\mathbb{Z}$. If $\mathbb{F}$ is algebraically closed & of char 0, then we get a bijection between the irreducible representations of G and the mth roots of 1. But to identify $\{z \mid z^m=1\}$ w. $\mathbb{Z}/m\mathbb{Z}$ one needs to choose a primitive root of 1. And there's no canonical choice.

2.1) Binary dihedral groups

Here we revisit the binary dihedral group from Problem 2 in HW1. We'll classify its irreducible representations using the general theory and compute the characters.

Recall that we are dealing with the group

$$G=\left\{\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}, \begin{pmatrix}0 & \varepsilon\\ -\varepsilon^{-1} & 0\end{pmatrix} \middle| \varepsilon^{2n}=1\right\} \subset GL_2(\mathbb{C}).$$

Let's compute the number of conjugacy classes. We have the normal commutative subgroup of index 2: $H=\left\{\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}\right\}$

So the conjugacy classes are of two kinds:

- contained in H, those are $\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, s\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}s^{-1}=\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}\right\}$, where $s=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. So we have $n-1$ classes w. two elements ($\lambda\neq\pm1$), and two classes w. one element each: $\lambda=\pm1$.

- not contained in H. An exercise is to show that there are two: $\left\{\begin{pmatrix}0 & \lambda\\ -\lambda^{-1} & 0\end{pmatrix} \middle| \lambda^n=1\right\}$, $\left\{\begin{pmatrix}0 & \lambda\\ -\lambda^{-1} & 0\end{pmatrix} \middle| \lambda^n=-1\right\}$.

The total number of conjugacy classes is $(n+1)+2=n+3$

The next step is to classify the 1-dimensional representations. For this we need to compute $G/(G,G)$, as the 1-dimensional representations of this group are in bijection w. those for G (Sec 1.1 of Lec 5). For our purposes, it will be enough to compute $|G/(G,G)|$. Note that

$$s\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}s^{-1}\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}=\begin{pmatrix}\lambda^{-2} & 0\\ 0 & \lambda^{2}\end{pmatrix}$$

So $\left\{\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix} \middle| \varepsilon^n=1\right\}\subset(G,G)$.

One can show that we get an equality. Alternatively, let $U_1,\dots,U_{n+3}$ be the irreducible representations. Then we know that

$$(*) \qquad \sum_{i=1}^{n+3} (\dim U_i)^2 = 4n.$$

Since $|G/(G,G)| \leq 4$, we have at most 4 representations of $\dim = 1$. And then we have $\geq n-1$ representations of $\dim \geq 2$. Using $(*)$, we see that we have <u>exactly</u> 4 1-dimensional rep's and $n-1$ irreducible representations of dimension 2. This is what we've got in Prob 2 of HW1 with Linear algebra.

We now explain how to compute the characters of 2-dimensional irreducible representations. For this we need a more detailed information about 2-dimensional irreducible representations from Problem 2.

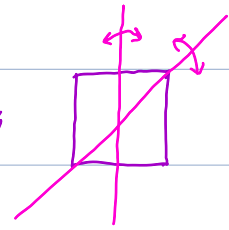
Recall that each such representation has basis v, sv where the operators s and t act by

$$\begin{pmatrix} 0 & \lambda^m \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where $\lambda^{2m} = 1$ & $\lambda \neq \{\pm 1\}$. From here we see that

$$\chi_V(t^i) = \lambda^i + \lambda^{-i}, \ \chi_V(st^i) = 0 \quad \forall\ i = 0, \dots, 2n-1.$$

Side Remark: The quotient G by $\{\pm I\} = \{1, t^n\}$ a central subgroup is identified w. the usual dihedral group: the symmetries of the regular n-gon.



These symmetries are of the following form:

- Rotations around the center by $\frac{2\pi k}{n}$, $k = 0, \dots n-1$
- Reflections about lines passing through opposite vertices & midpoints of opposite sides (n is even) or through a vertex & midpoint of the opposite side (n is odd).

Under an identification of $G/\{1, t^n\}$ w. the dihedral group t goes to the rotation by $\frac{2\pi k}{n}$ for any k coprime to n & s goes to any of the reflections (exercise) – so there are a lot of choices.

3) More orthogonality!

There are other orthogonality results. For simplicity, assume $\mathbb{F} = \mathbb{C}$. The following is Theorem 4.5.4 in [E]

For $g \in G$, let $Z_G(g) := \{h \in G \mid hg = gh\}$, the "centralizer."

Proposition 1: Let $U_1, \dots U_k$ be the complete list of irreducible representations of G (up to isomorphism, so that $k = \#$ conjugacy classes in G). Then, for $g, h \in G$

$$\sum_{i=1}^{k} \chi_V(g)\overline{\chi_V(h)} = \begin{cases} |Z_G(g)|, & \text{if } g \,\&\, h \text{ are conjugate} \\ 0, & \text{else} \end{cases}$$

This is a consequence of our main orthogonality theorem, see Remark 4.5.5 in [E].

Another orthogonality result concerns the matrix coefficients of irreducible representations. Note that the Hermitian scalar product $\langle \cdot, \cdot \rangle$: $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$, makes sense of $\mathrm{Fun}(G, \mathbb{C})$. On the other hand, by Sec 1.3 in Lec 6, every (finite dimensional) representation of G admits a G-invariant Hermitian scalar product. Let $\langle \cdot, \cdot \rangle_{U_\ell}$ denote such a product on U_ℓ, $\ell = 1, \dots k$, one of the irreducible representations. Pick an orthonormal basis $u_{\ell,1}, \dots u_{\ell,d_\ell}$ ($d_\ell := \dim U_\ell$) w.r.t. $\langle \cdot, \cdot \rangle_{U_\ell}$.

Then we have functions (matrix coefficients)

$$f_\ell^{ij}(g) := \langle g v_i, v_j \rangle_{U_\ell} : G \to \mathbb{C}, \quad \ell = 1, \dots, k, \quad i,j = 1, \dots, d_\ell$$

Proposition 2: We have $\langle f_\ell^{ij}, f_{\ell'}^{i'j'} \rangle = \delta_{ii'} \delta_{jj'} \delta_{\ell\ell'} / \dim U_\ell$

Moreover, the elements f_ℓ^{ij} form a basis in $\mathrm{Fun}(G, \mathbb{C})$.

For a proof, see Proposition 4.7.1 in [E].