Lecture 11, Characters 4.

1) Applications of orthonormality.
2) Representations of direct products.
3) What's next: values of characters & applications.
4) Bonus: Grothendieck ring.

Ref: [E], Secs 4.9, 5.6, 5.7.

1) Applications of orthonormality.

We continue to explore the applications of the theorem on the orthogonality of characters from Lec 9. Recall that the theorem says:

Let $\mathbb{F}$ be an algebraically closed field of characteristic 0 & G a finite group. Then the characters of irreducible representations of G form an orthonormal basis in $Cl(G)$ w.r.t. to the form $(\cdot,\cdot)$:

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1})$$

Application 1: if U, V are finite dimensional representations of G. TFAE:

(a) $U \simeq V$ (isomorphic)

(b) $\chi_U = \chi_V$.

Proof:

(a) $\Rightarrow$ (b): was proved in Sec 2 of Lec 8.

(b) $\Rightarrow$ (a): By Maschke's thm, U & V are completely reducible. So if $U_1, \ldots, U_k$ be all irreducible representations, then $U \simeq \bigoplus_{i=1}^{k} U_i^{\oplus m_i}$, $V \simeq \bigoplus_{i=1}^{k} U_i^{\oplus n_i}$ for some $m_i, n_i \in \mathbb{Z}_{\geq 0}$. Recall that for two representations U', V' we have

$$\chi_{U' \oplus V'} = \chi_{U'} + \chi_{V'},$$

Lemma in Sec 2.2 of Lec 8. So $\chi_U = \sum_{i=1}^{k} m_i \chi_{U_i}$ & $\chi_V = \sum_{i=1}^{k} n_i \chi_{U_i}$. By the theorem, $\chi_{U_1}, \ldots \chi_{U_k}$ are linearly independent. So $\chi_U = \chi_V \Leftrightarrow m_i = n_i \; \forall i \Leftrightarrow U \simeq V$. $\square$

Remark: We've used linear independence & Maschke's Thm not orthogonality. Note also that Application 1 implies Fact from Sec 2 of Lec 8.

Application 2: In the above notation, the multiplicity n_i of U_i in V is (χ_{U_i}, χ_V):

$$(\chi_{U_i}, \chi_V) = (\chi_{U_i}, \sum_{j=1}^{k} n_j \chi_{U_j}) = [(\chi_{U_i}, \chi_{U_j}) = \delta_{ij}] = n_i.$$

This also yields another proof of Application 1.

Example 1: Let $V = \mathbb{F}G$ so that, Sec 2.1 of Lec 8, we have $\chi_{\mathbb{F}G} = |G|\delta_e$. Then $(\chi_{U_i}, \chi_{\mathbb{F}G}) = \frac{1}{|G|}\sum_{g\in G} \chi_{U_i}(g)|G|\delta_e(g)$ $= \chi_{U_i}(e) = tr(Id_{U_i}) = \dim U_i$. We recover (3) from Sec 2.1 of Lec 7.

Example 2: We can apply $n_i = (\chi_{U_i}, \chi_V)$ to compute the decompositions of tensor products into irreducibles. Recall, Sec 1.5 of Lec 9, that $\chi_{U_j\otimes U_k} = \chi_{U_j}\chi_{U_k}$ (see also Addendum to Lec 9 for a different proof) $\Rightarrow$ multiplicity of U_i in $U_j \otimes U_k$ is $(\chi_{U_i}, \chi_{U_j}\chi_{U_k})$.

For example, take $G = S_4$, $U_j = U_k = \mathbb{F}_0^4$. Recall the character table of S_4 from Sec 2.2 of Lec 8. We also add one more row, for $\mathbb{F}_0^4 \otimes \mathbb{F}_0^4$, & the number of elements in conjugacy classes.

	$\#=1$	$\#=6$	$\#=3$	$\#=8$	$\#=6$
	$1+1+1+1$	$2+1+1$	$2+2$	$3+1$	4
triv	1	1	1	1	1
$\mathbb{F}_0^4$	3	1	-1	0	-1
V_2	2	0	2	-1	0
$sgn\otimes\mathbb{F}_0^4$	3	-1	-1	0	1
sgn	1	-1	1	1	-1
$\mathbb{F}_0^4\otimes\mathbb{F}_0^4$	9	1	1	0	1

Note that g, g^{-1} have the same cycle type hence conjugate for all $g\in S_4$, so $(f_1, f_2)=\frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g)$.

For example, the multiplicity of $\mathbb{F}_0^4$ in $\mathbb{F}_0^4\otimes\mathbb{F}_0^4$ is:

$$\frac{1}{|G|}\sum_{g\in G}\chi_{\mathbb{F}_0^4}(g)\chi^2_{\mathbb{F}_0^4}(g)=[\chi:=\chi_{\mathbb{F}_0^4}]=\frac{1}{24}\left(\chi(1)^3\cdot 1+\chi((12))^3\cdot 6+\chi((12)(34))^3\cdot 3+\chi((123))^3\cdot 8+\chi((1234))^3\cdot 6\right)=$$

$$=\frac{1}{24}\left(27\cdot 1+1\cdot 6+(-1)\cdot 3+0\cdot 8+(-1)\cdot 6\right)=1.$$

Exercise: Prove that $\mathbb{F}_0^4\otimes\mathbb{F}_0^4=\mathbb{F}_0^4\oplus triv\oplus V_2\oplus sgn\otimes\mathbb{F}_0^4$.

Application 3: Detecting characters of irreducibles - will become important later.

Let $f \in Cl(G)$ be of the form $\sum_{i=1}^{k} n_i \chi_{U_i}$, where $U_1, \dots U_k$ are different irreducibles of G & $n_i \in \mathbb{Z}$. TFAE.

(a) f is a character of an irreducible.

(b) $(f,f)=1$ & $f(e)>0$.

Proof: We prove (b) $\Rightarrow$ (a), leaving (a) $\Rightarrow$ (b) as an exercise.

Since χ_{U_i}'s are orthonormal, we have $(f,f)=\sum_{i=1}^{k} n_i^2$. From $(f,f)=1$, we deduce that $f=\pm\chi_{U_i}$ for some i. And $\chi_{U_i}(e)=\dim U_i$ is positive. This implies (a). $\square$

Remark: Let's comment on an ideological point. The study of characters reduces questions about representations (group homomorphisms, that can be hard) to questions about characters (functions, that can be easier).

2) Representations of direct products.

Let $\mathbb{F}$ be an algebraically closed field w. $\operatorname{char}\mathbb{F}=0$. Let G_1, G_2 be finite groups. We want to relate the irreducible representations of $G_1\times G_2$ to those of G_1, G_2.

Let V_i be a representation of G_i, $i=1,2$. We view V_i as a representation of $G_1\times G_2$ via pullback under the projection $G_1\times G_2\to G_i$, explicitly $(g_1,g_2)v_i := g_i v_i$. And then $V_1\otimes V_2$ is a representation of $G_1\times G_2$ (as a tensor product) with $(g_1,g_2).v_1\otimes v_2 = g_1v_1\otimes g_2v_2$.

Theorem: the irreducible representations of $G_1\times G_2$ are exactly of the form $V_1\otimes V_2$, where V_i is an irreducible of G_i, $i=1,2$.

Proof:

Step 1: Check that $V_1\otimes V_2$ is irreducible. We'll do this by computing the character and using Application 3. We'll also see that if V_1', V_2' are irreducibles, then $V_1\otimes V_2 \simeq V_1'\otimes V_2' \Rightarrow V_i\simeq V_i'$, $i=1,2$.

$\chi_{V_1\otimes V_2}(g_1,g_2) = \chi_{V_1}(g_1,g_2)\chi_{V_2}(g_1,g_2) = [(g_1,g_2)$ acts on V_i via $g_i]$ $= \chi_{V_1}(g_1)\chi_{V_2}(g_2)$. So

$$(\chi_{V_1\otimes V_2}, \chi_{V_1'\otimes V_2'}) = \frac{1}{|G_1||G_2|}\sum_{g_1\in G_1}\sum_{g_2\in G_2}\chi_{V_1}(g_1)\chi_{V_2}(g_2)\chi_{V_1'}(g_1^{-1})\chi_{V_2'}(g_2^{-1})$$

$$= \frac{1}{|G_1|} \sum_{g_1 \in G_1} \chi_{V_1}(g_1)\chi_{V_1'}(g_1^{-1}) \cdot \frac{1}{|G_2|} \sum_{g_2 \in G_2} \chi_{V_2}(g_2)\chi_{V_2'}(g_2^{-1}) = (\chi_{V_1}, \chi_{V_1'})(\chi_{V_2}, \chi_{V_2'})$$

$$= \begin{cases} 1, & V_1 \simeq V_1' \ \& \ V_2 \simeq V_2' \\ 0, & \text{else} \end{cases}$$

In the first case we conclude that $V_1 \otimes V_2$ is irreducible using Application 3. By the 2nd case, if (V_1, V_2) is different from (V_1', V_2'), then $(\chi_{V_1 \otimes V_2}, \chi_{V_1' \otimes V_2'}) = 0$. On the other hand, $V_1 \otimes V_2 \simeq V_1' \otimes V_2'$ $\Rightarrow \chi_{V_1 \otimes V_2} = \chi_{V_1' \otimes V_2'}$, so, by case 1, $(\chi_{V_1 \otimes V_2}, \chi_{V_1' \otimes V_2'}) = 1$. This contradiction shows that $V_1 \otimes V_2 \simeq V_1' \otimes V_2' \Rightarrow V_1 \simeq V_1', V_2 \simeq V_2'$.

Step 2: We show that there are no other irreducibles. Let k_i be the number of conjugacy classes in G_i. Then the number of conjugacy classes in $G_1 \times G_2$ is $k_1 k_2$ (exercise). We have k_i pairwise non-isomorphic irreducible representations of G_i. Step 1 yields $k_1 k_2$ pairwise non-isomorphic irreducible representations of $G_1 \times G_2$. Since the number of irreducibles is the number of conjugacy classes, there are indeed no other irreducibles $\square$

3) What's next: values of characters & applications.

Here we assume that $\mathbb{F}=\mathbb{C}$.

Characters of irreducibles are functions on G. One can ask what their possible values are. We will state some results now & prove them later.

Here's an easy consideration. Let V be a finite dimensional representation of G. In the proof of Lemma in Sec 1.3 of Lec 10 we have pointed out that $\forall g\in G$, the eigenvalues of g_V are roots of unity. One can show that this implies that $\chi_V(g)$, their sum, is an "algebraic integer" - we will give a definition in the next lecture.

Here's a more interesting result in the same spirit. Let U be an irreducible representation of G, and $g\in G$. Let C be the conjugacy class of g (in G).

Proposition: The number $\dfrac{|C|\chi_U(g)}{\dim U}$ is an algebraic integer.

These considerations have a number of important consequences that we are going to cover. The first is the Frobenius divisibility theorem:

Theorem 1: Let U be an irreducible representation of G. Then $\dim U$ divides $|G|$.

The 2nd application is the Burnside theorem, see Sec 4 of Lec 1.

Theorem 2: Let p, q be primes, and $a, b \in \mathbb{Z}_{\geqslant 0}$. A group of order $p^a q^b$ cannot be simple.

4) Bonus: Grothendieck ring (prereq MATH 380)

Recall, Sec 1.4 of Lec 4, that vector spaces (and hence group representations) behave like elements of a commutative ring w.r.t. the operations of $\oplus$ & $\otimes$. In this section we formalize this.

Let $\mathbb{F}$ be a field & G be a group. Let $\operatorname{Rep} G$ denote the category of fin. dimensional representations of G. We define the abelian group $K_0(\operatorname{Rep} G)$ (the Grothendieck group) as

- the quotient of the free group generated by symbols $[U]$, one for each representation U up to isomorphism,
- modulo the relations $[U]+[W]=[V]$ for short exact sequences $0 \to U \to W \to V \to 0$.

Exercise 1: Let G be a finite group. Assume that the number of finite dimensional irreducible representations of G is finite (which we know when $\operatorname{char} \mathbb{F} \nmid |G|$ and which is true in general). Let $U_1, \dots U_k$ be these representations. Then $K_0(\operatorname{Rep} G)$ is a free abelian group w. basis $[U_i], i=1,\dots k$.

Exercise 2: Show that there is a unique commutative associative ring structure on $K_0(\mathrm{Rep}\, G)$ s.t. $[U][V] = [U \otimes V]$. Moreover, $[\mathrm{triv}]$ is the unit.

Exercise 3: Show that $[V] \to \chi_V$ is a well-defined ring homomorphism $K_0(\mathrm{Rep}\, G) \longrightarrow Cl(G)$ that is viewed as a ring w.r.t. addition and multiplication of functions.

Moreover, if $\mathbb{F}$ is algebraically closed of characteristic 0, then this homomorphism induces an isomorphism

$$\mathbb{F} \otimes_{\mathbb{Z}} K_0(\mathrm{Rep}\, G) \xrightarrow{\sim} Cl(G).$$

Remarks: 1) The class $[V]$ of V in $K_0(\mathrm{Rep}\, G)$ can be viewed as the "universal character" - it incorporates all information about a representation that is insensitive to different extensions of the same two representations. If $K_0(\mathrm{Rep}\, G) \longrightarrow Cl(G)$ is injective, it means that χ_V captures all information about V that is insensitive to extensions.

2) Classification of algebraic structures by means of elements of another algebraic structure is a common theme in Algebra. Class groups (of Dedekind domains) is one example. We'll see another example: the Brauer group of a field later in our study of central simple algeb-ras.