Lecture 13, Applications of characters.

1) Integrality properties.

2) Burnside theorem.

Ref: Secs 5.3, 5.4 in [E].

1) Integrality properties.

In Section 3 of Lec 11, we have mentioned that various numbers related to characters are algebraic integers. Here we state & prove these results and give a quick application, the Frobenius divisibility theorem.

Recall that the base field is $\mathbb{C}$, and G is a finite group.

1.1) Results.

Lemma: Let V be a finite dimensional representation of G. Then $\chi_V(g) \in \overline{\mathbb{Z}}$ $\forall g \in G$.

Proof: As was noted in the proof of Lemma in Sec 1.3 of Lec 10, all eigenvalues of g_V are roots of unity (g, hence g_V has finite order) hence are in $\overline{\mathbb{Z}}$. By Proposition in Sec 2 of Lec 12, their sum, which is $\chi_V(g)$, is in $\overline{\mathbb{Z}}$. $\square$

Here's a more subtle result.

Proposition: Let U be an irreducible representation of G, $g\in G$ & $C\subset G$ be the conjugacy class of g. Then $\frac{|C|\chi_U(g)}{\dim U}\in\overline{\mathbb{Z}}$.

We'll postpone a proof & discuss an application first.

1.2) Frobenius divisibility.

Thm: In the notation of Proposition, $|G|/\dim U\in\mathbb{Z}$.

Proof: The idea is to show that $\frac{|G|}{\dim U}$ $(\in\mathbb{Q})$ is in $\overline{\mathbb{Z}}$. Then we use $\mathbb{Q}\cap\overline{\mathbb{Z}}=\mathbb{Z}$ (Corollary in Sec 1.1 of Lec 12).

By Application 3, Sec 1 of Lec 11, $(\chi_U,\chi_U)=1$, i.e

$$\frac{1}{|G|}\sum_{g\in G}\chi_U(g)\chi_U(g^{-1})=1 \qquad (1)$$

Let $C_1, \ldots C_k$ be the conjugacy classes in G & $g_i\in C_i$ so that $\chi_U(g)=\chi_U(g_i)$, $\chi_U(g^{-1})=\chi_U(g_i^{-1})$ $\forall g\in C$ (note that g^{-1} is conjugate to g_i^{-1} b/c $(hgh^{-1})^{-1}=hg^{-1}h^{-1}$). So

$$\sum_{g\in G}\chi_U(g)\chi_U(g^{-1})=\sum_{i=1}^{k}\left(|C_i|\chi_U(g_i)\right)\chi_U(g_i^{-1}).$$

Multiplying by (1) by $\frac{|G|}{\dim U}$, we get

$$\sum_{i=1}^{k}\frac{|C_i|\chi_U(g_i)}{\dim U}\cdot\chi_U(g_i^{-1})=\frac{|G|}{\dim U} \qquad (2)$$

By Lemma (resp. Proposition) from Sec 1.1 in Lec 12, the 2nd (resp. the 1st) factor is in $\overline{\mathbb{Z}}$. Since $\overline{\mathbb{Z}}$ is closed under sums & products, the l.h.s. of (2) is in $\overline{\mathbb{Z}}$. So, $\frac{|G|}{\dim U}\in\overline{\mathbb{Z}}$, and we are done $\square$

1.3) Proof of Proposition.

By Lemma in Sec 1.3, any element $z\in Z(\mathbb{C}G)$ acts on U via $\frac{\chi_U(z)}{\dim U}Id_U$. Since for $z,z'\in Z(\mathbb{C}G)$, we have $(zz')_U=z_U z'_U$, it follows that $\rho: z\mapsto\frac{\chi_U(z)}{\dim U}: Z(\mathbb{C}G)\longrightarrow\mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism.

We need a particular choice of z: $z_C = \sum_{h \in C} h \in Z(\mathbb{C}G)$ so that $\chi_U(z_C) = |C|\chi_U(g)$ $(g \in C)$. Our goal is therefore to prove

Lemma: $\rho(z_C) \in \overline{\mathbb{Z}}$.

Proof: In the proof we'll need a subring

$$\mathbb{Z}G = \{\sum_{g \in G} a_g g \mid a_g \in \mathbb{Z}\} \subset \mathbb{C}G$$

Note that $\mathbb{Z}G$ is a finitely generated free abelian group (w.r.t $+$). Observe that $z_C \in \mathbb{Z}G$.

The scheme of the proof is as follows:

(i) we show that $\mathrm{Span}_{\mathbb{Z}}(z_C^i \mid i \geqslant 0) \subset Z(\mathbb{C}G)$ is fin. generated as an abelian group.

(ii) deduce that $\mathrm{Span}_{\mathbb{Z}}(\rho(z_C)^i \mid i \geqslant 0) \subset \mathbb{C}$ is fin. generated.

(iii) conclude that $\rho(z_C) \in \overline{\mathbb{Z}}$.

Check (i): Note that, since $z_C \in \mathbb{Z}G$ & $\mathbb{Z}G$ is a subring, we get $\mathrm{Span}_{\mathbb{Z}}(z_C^i) \subset \mathbb{Z}G$. By Fact in Sec 1.2 of Lec 12, $\mathrm{Span}_{\mathbb{Z}}(z_C^i)$ is fin. generated since $\mathbb{Z}G$ clearly is.

Check (ii): $\mathrm{Span}_{\mathbb{Z}}(\rho(z_C)^k) = [\rho(z_C)^k = \rho(z_C^k)] = \rho(\mathrm{Span}_{\mathbb{Z}}(z_C^k))$

Since ρ is, in particular, a homomorphism of abelian groups, we use (i) to deduce that $\mathrm{Span}_{\mathbb{Z}}(\rho(z_C)^k)$ is finitely generated.

Check (iii): follows from Proposition in Sec 1.2 of Lec 12 $\square$

2) Burnside theorem.

Our job now is to prove:

Theorem (Burnside): A group of order $p^a q^b$ cannot be simple

The proof we'll be based on the following proposition:

Proposition: Let G be a finite group, $C \subset G$ a conjugacy class, and U be an irreducible representation of G. Assume $GCD(\dim U, |C|) = 1$. Then one of the following holds:

(a) $\chi_U(g) = 0$ for $g \in C$

(b) g_U is scalar $\forall g \in C$.

2.1) How Theorem follows.

We first deduce the theorem from Proposition and then prove the proposition.

Corollary (of Proposition) Suppose G is simple, U is non-trivial & $g \neq e$. Under the assumptions of Proposition, (a) holds.

Proof: Let ρ be the homomorphism $G \to GL(U)$. Then $\ker \rho$ is a normal subgroup. We have $\ker \rho \neq G$ b/c U is non-trivial. Since G is simple, $\ker \rho = \{e\}$, so we can view G as a subgroup of $GL(U)$. Any subgroup of scalar operators is normal in $GL(U)$ (scalars commute w. every operator). It follows that $H := \{h \in G \mid h_U \text{ is scalar}\}$ is normal. G is not abelian, while H is, so $H \neq G$. So, $H = \{e\}$ and (a) holds. $\square$

Proof of Theorem:

Step 0 (reduction to $a, b > 0$): a p-group has nontrivial center so cannot be simple. Hence both a & b are positive.

Step 1 ($\exists$ conj. class $C \neq \{e\}$ w. $|C|$ = prime power). Let $C_1 = \{e\}, C_2, \ldots, C_k$ be the conjugacy classes in G. We have $\sum_{i=2}^{k} |C_i| = |G| - |C_1| = |G| - 1 = p^a q^b - 1$, not divisible by pq. So $\exists i$ s.t. $|C_i|$ is not divisible by pq. Since $|C_i| \mid |G|$, $|C_i|$ must be a prime power, say p^c ($c \geq 0$). Pick $g \in C_i$.

Step 2: Recall that $\chi_{\mathbb{C}G}(g) = 0$ if $g \neq e$, Example in Sec 2.1 of Lec 8. Also, let $U_1, U_2, \ldots U_k$ are the irreducible representations w. U_1 = triv. By Theorem in Sec 2.1 in Lec 7,

$$\mathbb{C}G = \bigoplus_{i=1}^{k} U_i^{\oplus \dim U_i} \Rightarrow \chi_{\mathbb{C}G} = \sum_{i=1}^{k} (\dim U_i) \cdot \chi_{U_i} \Rightarrow$$

$$0 = \chi_{\mathbb{C}G}(g) = \sum_{i=1}^{k} (\dim U_i) \chi_{U_i}(g) \Rightarrow$$

$$-1 = \sum_{i=2}^{k} (\dim U_i) \chi_{U_i}(g) \qquad (*)$$

Step 3: Let $U_2, \ldots U_\ell$ be the irreducibles w. dim coprime to p, & $U_{\ell+1}, \ldots U_k$ be those w. $\dim U_i \,\vdots\, p$. By Proposition, for $i = 2, \ldots \ell$, $\chi_{U_i}(g) = 0$ b/c the conjugacy class C_i of g has p^c elements so coprime to $\dim U_i$. We can rewrite $(*)$ as

$$-\frac{1}{p} = \sum_{i=\ell+1}^{k} \frac{\dim U_i}{p} \chi_{U_i}(g)$$

Note that $\frac{\dim U_i}{p} \in \mathbb{Z}$; $\chi_{U_i}(g) \in \overline{\mathbb{Z}}$ by Lemma in Sec 1.1. So by Proposition in Sec 2 of Lec 12, the r.h.s. is in $\overline{\mathbb{Z}}$. So $-\frac{1}{p} \in \overline{\mathbb{Z}}$. And $\overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$ (Corollary in Sec 1.1 of Lec 12). So $-\frac{1}{p} \in \mathbb{Z}$, a contradiction $\square$

2.2) Proof of Proposition.

We'll prove this proposition modulo a lemma.

Lemma: Let $\varepsilon_1, \dots \varepsilon_n$ be roots of 1. If $z := \frac{\varepsilon_1 + \dots + \varepsilon_n}{n} \in \overline{\mathbb{Z}}$, then either $\varepsilon_1 = \dots = \varepsilon_n$ or $\varepsilon_1 + \dots + \varepsilon_n = 0$.

Proof of Proposition:

We know that $\chi_U(g) \in \overline{\mathbb{Z}}$ (Lemma in Sec 1.1) & $\frac{|C| \chi_U(g)}{\dim U} \in \overline{\mathbb{Z}}$ (Proposition in Sec 1.1). Since $GCD(|C|, \dim U) = 1$, we have $r|C| + s \dim U = 1$ for some $r, s \in \mathbb{Z}$, hence

$$\frac{\chi_U(g)}{\dim U} \in \overline{\mathbb{Z}}$$

Let $\varepsilon_1, \dots \varepsilon_n$ be the eigenvalues of g_U ($n = \dim U$) so that $\chi_U(g) = \varepsilon_1 + \dots + \varepsilon_n$. By Lemma, either $\varepsilon_1 + \dots + \varepsilon_n = 0$ (in which case $\chi_U(g) = 0$) or

$\xi = \xi = \ldots = \xi_n$. Note that g_u has finite order, so cannot have Jordan blocks of size >1. It follows that $g_u = \mathrm{diag}(\xi, \varepsilon_2, \ldots \xi_n)$, constant $\square$