Lecture 15, Induced representations.

0) Recap & plan

1) Proof of Frobenius reciprocity.

2) Character formula for the induced representation.

Ref: Secs 5.8-5.10 in [E]

0) Recap & plan

Let $\mathbb{F}$ be a field, $H\subset G$ be finite groups and U be a (finite dimensional) representation of H. In Sec 1.1 of Lec 14 we have constructed the induced representation of G:

$$\operatorname{Ind}_H^G U \,(= \operatorname{Map}_H(G,U)) = \{\varphi: G\longrightarrow U \mid \varphi(gh^{-1}) = h_U\varphi(g)\}$$

w. $[g_1.\varphi](g) = \varphi(g_1^{-1}g)$.

In Sec 2.1 of Lec 14, we have stated the important property, the Frobenius reciprocity:

for (finite dimensional) representations U of H & V of G we have a natural vector space isomorphism

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G U) \xrightarrow{\sim} \operatorname{Hom}_H(V, \underbrace{U}_{=\operatorname{Res}_H^G U}) \qquad (*)$$

The plan for this lecture is as follows:

- Prove Frobenius reciprocity.
- State & prove the Frobenius formula for the characters of induced representations.
- Make curious observations about representations of the form $\operatorname{Ind}_{S_\lambda}^{S_n} \mathrm{triv}$ & $\operatorname{Ind}_{S_\lambda}^{S_n} \mathrm{sgn}$ for $n=4$ that will be generalized to arbitrary n after the break leading to the classification of irreducible representations of S_n.

1) Proof of Frobenius reciprocity.

- First, we construct a map in (*). Define the map

$$ev: \operatorname{Ind}_H^G U = \operatorname{Map}_H(G,U) \longrightarrow U,\ ev(\varphi):=\varphi(e)$$

We claim that it is a homomorphism of representations of H:

$ev(h.\varphi) = [h.\varphi](e) = \varphi(h^{-1}) = [\varphi \text{ is equivariant}] = h_U.\varphi(e)$.

Now our map (*) is $\psi \mapsto ev\circ\psi$. Since ev & ψ are homomorphisms of representations of H, so is their composition. Hence we indeed get a well-defined linear map in (*).

- Now we are going to construct an inverse. Let $\eta\in$

$\mathrm{Hom}_H(V,U)$. We are going to define $\psi_\eta : V \to \mathrm{Map}_H(G,U)$. We do this by $[\psi_\eta(v)](g) := \eta(g_V^{-1}v)$. We need to check that

- $\psi_\eta(v) \in \mathrm{Map}_H(G,U) \Leftrightarrow [\psi_\eta(v)](gh^{-1}) = h_U([\psi_\eta(v)](g))$.

$[\psi_\eta(v)](gh^{-1}) = \eta(h_V g_V^{-1} v) = h_U\, \eta(g_V^{-1}v) = h_U([\psi_\eta(v)](g))$ ✓

- ψ_η is G-equivariant: $\psi_\eta(g'_V v) = g'_{\mathrm{Map}_H(G,U)}\, \psi_\eta(v)$, $\forall g' \in G$.

$\psi_\eta(g'_V v)(g) = \eta(g_V^{-1} g'_V v)$

$[g'_{\mathrm{Map}_H(G,U)}\, \psi_\eta(v)](g) = [\psi_\eta(v)](g'^{-1}g) = \eta(g_V^{-1} g'_V v)$ - checked.

- We need to show that $\eta \mapsto \psi_\eta$ & $\psi \mapsto ev \circ \psi$ are inverse to each other: $ev \circ \psi_\eta = \eta$ & $\psi_{ev\circ\psi} = \psi$.

$ev \circ \psi_\eta(v) = [\psi_\eta(v)](e) = \eta(v)$ ✓.

$[\psi_{ev\circ\psi}(v)](g) = ev(\psi(g_V^{-1}v)) = [\psi(g_V^{-1}v)](e) = [\psi \text{ is } G\text{-equiv't}]$
$= [\psi(v)](g) \Rightarrow \psi_{ev\circ\psi} = \psi$ $\square$

2) Character formula for the induced representation.

2.1) Main result.

Let g_i, $i=1,\dots,\ell$, be representatives of the cosets in G/H, so that $\ell = |G/H|$. Every element of G is uniquely written as $g_i h^{-1}$, $i=1,\dots,\ell$, $h \in H$.

Theorem (Frobenius)

$$\chi_{\mathrm{Ind}_H^G U}(g) = \sum_{i \mid g_i^{-1} g g_i \in H} \chi_U(g_i^{-1} g g_i) \qquad (1)$$

Proof: For $i = 1, \dots, \ell$, let

$$\mathrm{Map}_H(G,U)_i = \{\varphi \in \mathrm{Map}_H(G,U) \mid \varphi(g_j h^{-1}) = 0 \text{ if } j \neq i\}$$

We have $\mathrm{Map}_H(G,U)_i \xrightarrow{\sim} U$, $\varphi \mapsto \varphi(g_i)$

$$\& \ \mathrm{Map}_H(G,U) = \bigoplus_{i=1}^{\ell} \mathrm{Map}_H(G,U)_i$$

compare to Sec 1.2 in Lec 14.

Under the identification $\mathrm{Map}_H(G,U) = \bigoplus_{i=1}^{\ell} \mathrm{Map}_H(G,U)_i \xrightarrow{\sim} U^{\oplus \ell}$ we can view $g_{\mathrm{Map}_H(G,U)} \in \mathrm{End}(U^{\oplus \ell})$ as an $\ell \times \ell$-matrix $\alpha_{ij}(g)$ w. $\alpha_{ij}(g) \in \mathrm{End}(U)$: $g_{\mathrm{Map}_H(G,U)}(u_1, \dots, u_\ell) = \left(\sum_{j=1}^{\ell} \alpha_{ij}(g) u_j\right)_{i=1}^{\ell}$. So

$$\chi_{\mathrm{Ind}_H^G U}(g) = \sum_{i=1}^{\ell} \operatorname{tr} \alpha_{ii}(g). \qquad (2)$$

So we need to determine $\alpha_{ii}(g)$. Let $\varphi \in \mathrm{Map}_H(G,U)_i$. Then $[g.\varphi](g_i h^{-1}) = \varphi(g^{-1} g_i h^{-1})$. Note that

$$(\heartsuit) \quad g^{-1} g_i h^{-1} \in g_i H \Leftrightarrow g_i^{-1} g^{-1} g_i \in H \Leftrightarrow g_i^{-1} g g_i \in H.$$

If $(\heartsuit)$ doesn't hold, then $\alpha_{ii}(g) = 0$. If $(\heartsuit)$ holds then $g.\varphi \in \mathrm{Map}_H(G,U)_i$ & under the identification $\mathrm{Map}_H(G,U)_i \xrightarrow{\sim} U$,

H-equivariant

$$g.\varphi \mapsto [g.\varphi](g_i) = \varphi(g^{-1} g_i) = \varphi(g_i \overbrace{g_i^{-1} g^{-1} g_i}^{\in H}) = (g_i^{-1} g g_i)_U \varphi(g_i). \text{ So}$$

$\alpha_{ii}(g) = (g_i^{-1} g g_i)_U \Rightarrow \operatorname{tr} \alpha_{ii}(g) = \chi_U(g_i^{-1} g g_i) \Rightarrow$ r.h.s of (2) = r.h.s of (1) $\square$

2.2) Examples & applications.

Example 1: Suppose U is the 1-dimensional trivial representation. As we have seen in Sec 1.1 of Lec 14,

$$\mathrm{Ind}_H^G U \xrightarrow{\sim} \mathrm{Fun}(G/H, \mathbb{F})$$

By Sec 2.1 of Lec 8, $\chi_{\mathrm{Fun}(G/H,\mathbb{F})}(g) = \#\{x \in G/H \mid g.x = x\} = \#\{i \in \{1,\dots,\ell\} \mid g^{-1}g_i \in g_iH \Leftrightarrow g_i^{-1}gg_i \in H\}$. This agrees w. Thm.

Example 1': Let's use (1) to decompose $V := \mathrm{Ind}_H^G \mathrm{triv}$ w. $H = \{1, h\}$ into the direct sum of irreducibles (that also can be done $^{\text{so } h = h^{-1}}$ using the techniques of Sec 2.2 in Lec 14). Here we assume $\mathrm{char}\,\mathbb{F} = 0$ & $\mathbb{F}$ is algebraically closed. Since $\mathrm{char}\,\mathbb{F} = 0$, for general (H, U) the r.h.s. in (1) can be rewritten as

$\frac{1}{|H|} \sum_{k \in G \mid k^{-1}gk \in H} \chi_U(k^{-1}gk)$, (note that $g_i^{-1}gg_i \in H \Leftrightarrow k^{-1}gk \in H \ \forall k \in g_iH$)

Back to $H = \{1, h\}$ & $U = \mathrm{triv}$, note: $\chi_V(g) = 0$ unless $g = e$, in which case $\chi_V(e) = \dim V = \frac{1}{2}|G|$, or g is conjugate (in G) to h, where $\chi_V(g) = \chi_V(h) = \frac{1}{2}|\{k \in G \mid k^{-1}hk = h\}| = \frac{1}{2}|Z_G(h)|$.

Let W be an irreducible representation of G. Recall (see Application 2 in Sec 1 of Lec 11) that the multiplicity of

W in V equals $(\chi_W,\chi_V)=\frac{1}{|G|}\sum_{g\in G}\chi_W(g)\chi_V(g^{-1})$.

Let $C=\{ghg^{-1}|g\in G\}$ so that $|C|=|G|/|Z_G(h)|$.

Then $(\chi_W,\chi_V)=\frac{1}{|G|}(\underbrace{\chi_W(e)}_{\dim W}\frac{|G|}{2}+|C|\chi_W(h)\frac{1}{2}|Z_G(h)|)=$ product is $|G|$

$=\frac{1}{2}(\dim W+\chi_W(h))$.

Now apply this to $G=S_4$ & $H=S_2=\{e,(12)\}$. We use the character table for S_4 from Sec 2.2 of Lec 8 to conclude

$$\mathrm{Ind}_{S_2}^{S_4}\mathrm{triv}\simeq\mathrm{triv}\oplus(\mathbb{F}_0^4)^{\oplus 2}\oplus V_2\oplus \mathrm{sgn}\otimes\mathbb{F}_0^4.$$

And here's an application of Theorem.

Lemma (tensor identity): Suppose $\mathbb{F}$ is algebraically closed field of char 0. Let U & V be finite dimensional representations of H & G, respectively. We have

$$V\otimes\mathrm{Ind}_H^G U\xrightarrow{\sim}\mathrm{Ind}_H^G(\underbrace{V}_{\mathrm{Res}_H^G V}\otimes U),$$

an isomorphism of representations of G.

Proof: Recall, Application 1 in Sec 1 of Lec 11, that two representations are isomorphic if their characters are equal.

$\chi_{V\otimes \mathrm{Ind}_H^G U}(g) = $ [Sec 1.5 or Addendum of Lec 11] $= \chi_V(g)\, \chi_{\mathrm{Ind}_H^G U}(g)$

$\chi_{\mathrm{Ind}_H^G (V\otimes U)}(g) = \sum_{i \mid g_i^{-1} g g_i \in H} \chi_{V\otimes U}(g_i^{-1} g g_i) = \sum_{\dots} \chi_V(g_i^{-1} g g_i)\, \chi_U(g_i^{-1} g g_i)$

$= [\chi_V(g_i^{-1} g g_i) = \chi_V(g)] = \chi_V(g) \sum_{\dots} \chi_U(g_i^{-1} g g_i) = \chi_V(g)\, \chi_{\mathrm{Ind}_H^G U}(g)$ $\square$

Remark 1: The conclusion holds w/o restriction on $\mathbb{F}$ but the argument needs a few things from category theory. It's based on the following natural isomorphisms:

$\mathrm{Hom}_G(V', \mathrm{Ind}_H^G (V\otimes U)) \xrightarrow{\sim}$ [Frobenius reciprocity] $\mathrm{Hom}_H(V', V\otimes U)$ $\xrightarrow{\sim}$ [tensor-Hom adjunction] $\mathrm{Hom}_H(V'\otimes V^*, U) \xrightarrow{\sim}$ [Frobenius reciprocity] $\mathrm{Hom}_G(V'\otimes V^*, \mathrm{Ind}_H^G U) \xrightarrow{\sim}$ [tensor-Hom adjunction] $\mathrm{Hom}_G(V', V\otimes \mathrm{Ind}_H^G U)$.

Remark 2: Theorem has a fun application to the structure theory of finite groups. Let G be a finite group. By a Frobenius complement we mean a subgroup $H\subset G$ which is "as far from being normal as possible": $H\cap gHg^{-1} = \{e\}$ $\forall g\in G\setminus H$.

Consider the subset $K := \left(G \setminus \bigcup_{g\in G} gHg^{-1}\right) \sqcup \{e\}$. Then it is a subgroup (automatically normal) — this was proved by Frobenius.

2.3) Curious observations about $\mathrm{Ind}_{S_\lambda}^{S_4}\mathrm{triv}$, $\mathrm{Ind}_{S_\lambda}^{S_4}\mathrm{sgn}$.

Here's a complete list of decompositions of $\mathrm{Ind}_{S_\lambda}^{S_4}\mathrm{triv}$ into irreducibles. As before, $\mathbb{F}$ is algebraically closed & $\operatorname{char}\mathbb{F}=0$.

- $\lambda=(4)$: $\mathrm{Ind}_{S_4}^{S_4}\mathrm{triv}=\mathrm{triv}$
- $\lambda=(3,1)$: $\mathrm{Ind}_{S_3}^{S_4}\mathrm{triv}=\mathrm{Fun}(S_4/S_3,\mathbb{F})=[S_4/S_3 \xrightarrow{\sim}\{1,2,3,4\}$, exercise$]=\mathbb{F}^4=[$Lemma in Sec 1.2 of Lec 5$]=\mathrm{triv}\oplus\mathbb{F}_0^4$.

  permutation rep

- $\lambda=(2,2)$: $\mathrm{Ind}_{S_\lambda}^{S_4}\mathrm{triv}=\mathrm{triv}\oplus\mathbb{F}_0^4\oplus V_2$, Sec 2.1 of Lec 14.
- $\lambda=(2,1,1)$: $\mathrm{Ind}_{S_\lambda}^{S_4}\mathrm{triv}=\mathrm{triv}\oplus(\mathbb{F}_0^4)^{\oplus 2}\oplus V_2\oplus \mathrm{sgn}\otimes\mathbb{F}_0^4$, Example 1.
- $\lambda=(1,1,1,1)$: $\mathrm{Ind}_{\{e\}}^{S_4}=\mathrm{Fun}(G,\mathbb{F})=[$regular representation, see Thm in Sec 2.1 of Lec 7$]=\mathrm{triv}\oplus(\mathbb{F}_0^4)^{\oplus 3}\oplus V_2^{\oplus 2}\oplus(\mathrm{sgn}\otimes\mathbb{F}_0^4)^{\oplus 3}\oplus\mathrm{sgn}$.

We record this as a table, where rows correspond to $\mathrm{Ind}_{S_\lambda}^{S_4}\mathrm{triv}$, columns to the irreducibles & the entries are the multiplicities (we skip 0's)

Table 1: Decompositions of $\mathrm{Ind}_{S_\lambda}^{S_4}\,\mathrm{triv}$:

$\lambda \backslash V$	triv	$\mathbb{F}_0^4$	V_2	$\mathrm{sgn}\otimes\mathbb{F}_0^4$	sgn
4	1				
(3,1)	1	1			
(2,2)	1	1	1		
(2,1,1)	1	2	1	1	
(1,1,1,1)	1	3	2	3	1

Now we proceed to $\mathrm{Ind}_{S_\lambda}^{S_4}\,\mathrm{sgn}$, where by the sign representation of S_λ we mean the restriction of sgn from S_4. By Lemma in Sec 2.2 (tensor identity), we have $\mathrm{Ind}_{S_\lambda}^{S_4}\,\mathrm{sgn} \simeq \mathrm{sgn}\otimes \mathrm{Ind}_{S_\lambda}^{S_4}\,\mathrm{triv}$. So we get the following table

Table 2: Decompositions of $\mathrm{Ind}_{S_\lambda}^{S_4}\,\mathrm{sgn}$.

$\lambda \backslash V$	triv	$\mathbb{F}_0^4$	V_2	$\mathrm{sgn}\otimes\mathbb{F}_0^4$	sgn
(1,1,1,1)	1	3	2	3	1
(2,1,1)		1	1	2	1
(2,2)			1	1	1
(3,1)				1	1
(4)					1