Lecture 15, Induced representations.

0) Recap & plan

1) Proof of Frobenius reciprocity.

2) Character formula for the induced representation

Ref: Secs 5.8-5.10 in [E]
The plan for this lecture is as follows:

- Prove Frobenius reciprocity.
- State & prove the Frobenius formula for the characters of induced representations.
- Make curious observations about representations of the form $\text{Ind}_{S_n}^{S_{n+1}} \text{triv} \& \text{Ind}_{S_n}^{S_{n+1}} \text{sgn}$ for $n=4$ that will be generalized to arbitrary $n$ after the break leading to the classification of irreducible representations of $S_n$.

1) Proof of Frobenius reciprocity.

- First, we construct a map in (*). Define the map 
  \[ \text{ev}: \text{Ind}_{H}^{G} U = \text{Map}_{H}(\zeta, U) \to U, \text{ev}(\chi) = \chi(e) \]
  We claim that it is a homomorphism of representations of $H$:
  \[ \text{ev}(h \cdot \chi) = (h \cdot \chi)(e) = \chi(h^{-1}) = [\chi \text{ is equivariant}] = h \cdot \chi(e). \]
  Now our map (*) is $\chi \mapsto \text{ev} \circ \chi$. Since ev & $\chi$ are homomorphisms of representations of $H$, so is their composition. Hence we indeed get a well-defined linear map in (*).

- Now we are going to construct an inverse. Let $\psi \in$
Hom_H(V, U). We are going to define \( \psi_2: V \rightarrow \text{Map}_H(G, U) \). We do this by \( [\psi_2(v)](g) = \phi(g^{-1}v) \). We need to check that

\[
\begin{align*}
\bullet \quad \psi_2 \in \text{Map}_H(G, U) & \iff [\psi_2(v)](gh^{-1}) = h_u([\psi_2(v)](g)) \\
[\psi_2(v)](gh^{-1}) &= \phi(h_v g^{-1}v) = h_u \phi(g^{-1}v) = h_u([\psi_2(v)](g)) \quad \forall \quad g \in G,
\end{align*}
\]

\( \psi_2 \) is \( G \)-equivariant: \( \psi_2(g^\prime v) = g^{\prime \text{map}_H(G, U)} \psi_2(v), \forall g \in G \).

\( \psi_2(g^\prime v)(g) = \phi(g^{-1}g^\prime v) \)

\( [g^{\prime \text{map}_H(G, U)} \psi_2(v)](g) = [\psi_2(v)](g^{-1}g) = \phi(g^{-1}g^\prime v) \) - checked.

\( \bullet \quad \) We need to show that \( \phi \mapsto \psi_2 \) & \( \psi \mapsto \text{ev}_0 \psi \) are inverse to each other: \( \text{ev}_0 \psi = \phi \) & \( \psi \text{ev}_0 \psi = \psi \).

\( \text{ev}_0 \psi_2(v) = [\psi_2(v)](e) = \phi(v) \quad \forall \).

\( [\psi \text{ev}_0 \psi(v)](g) = \text{ev}(\phi(g^{-1}v)) = [\psi(g^{-1}v)](e) = [\psi \text{ is } G \text{-equiv} \text{r} \text{t}] \\
= [\psi(v)](g) \Rightarrow \psi \text{ev}_0 \psi = \psi \)

\[ \square \]

2) Character formula for the induced representation.

2.1) Main result.

Let \( g_i, i=1,...,\ell \), be representatives of the cosets in \( G/H \), so that \( \ell = |G/H| \). Every element of \( G \) is uniquely written as \( g_i h^{-1}, i=1,...,\ell, h \in H \).

3)
**Theorem (Frobenius)**

\[ X_{\text{Ind}_H^G} (g) = \sum_{i \in \mathcal{I}} X_U (g^{-1} g_i g) \]  

\( (1) \)

**Proof:** For \( i = 1, \ldots, l \), let

\[ \text{Map}_H (C, U)_i = \{ \varphi \in \text{Map}_H (C, U) \mid \varphi (g_i h^{-1}) = 0 \text{ if } j \neq i \} \]

We have \( \text{Map}_H (C, U)_i \xrightarrow{\sim} U, \varphi \mapsto \varphi (g_i) \)

\& \( \text{Map}_H (C, U) = \bigoplus_{i=1}^l \text{Map}_H (C, U)_i \)

compare to Sec. 1.2 in Lec 14.

Under the identification \( \text{Map}_H (C, U) = \bigoplus_{i=1}^l \text{Map}_H (C, U)_i \xrightarrow{\sim} U^\otimes l \)

we can view \( g_{\text{Map}_H (C, U)_i} \in \text{End} (U^\otimes l) \) as an \( l \times l \)-matrix \( d_{ij} (g) \)

w. \( d_{ij} (g) \in \text{End} (U) : g_{\text{Map}_H (C, U)_i} (u_1, \ldots, u_l) = (\sum_{j=1}^l d_{ij} (g) u_j)_{i=1}^l \). So

\[ X_{\text{Ind}_H^G} (g) = \sum_{i=1}^l \text{tr} \, d_{ii} (g). \]

\( (2) \)

So we need to determine \( d_{ii} (g) \). Let \( \varphi \in \text{Map}_H (C, U)_i \).

Then \( [g, \varphi] (g_i h^{-1}) = \varphi (g^{-1} g_i h^{-1}) \). Note that

\( (\bigcirc) \) \( g^{-1} g_i h^{-1} \in g_i H \iff g_i^{-1} g_i g : \in H \iff g_i^{-1} g_i g : H \).

If \( (\bigcirc) \) doesn't hold, then \( d_{ii} (g) = 0 \). If \( (\bigcirc) \) holds then

\[ g, \varphi \in \text{Map}_H (C, U)_i \] & under the identification \( \text{Map}_H (C, U)_i \xrightarrow{\sim} U, \]

\[ g, \varphi \mapsto [g, \varphi] (g_i) = \varphi (g^{-1} g_i) = \varphi (g_i g_i^{-1} g_i) = (g_i^{-1} g_i g) \varphi (g_i). \]

So

\[ d_{ii} (g) = (g_i^{-1} g_i, g_i) \Rightarrow \text{tr} \, d_{ii} (g) = X_U (g_i^{-1} g_i) \Rightarrow \text{r.h.s. of } (2) = \text{r.h.s. of } (1) \] \( \Box \)
2.2) Examples & applications.

Example 1: Suppose $U$ is the 1-dimensional trivial representation. As we have seen in Sec 1.1 of Lec 14,

$$\text{Ind}_H^G U \cong \text{Fun}(G/H, F)$$

By Sec 2.1 of Lec 8, $X_{\text{Fun}(G/H, F)}(g) = \{x \in G/H \mid g.x = x^g = g^{-1}x^g \} \cong \{i \in \{1, \ldots, n\} \mid g_i^* g_i \in g_i H \Leftrightarrow g_i^{-1} g_i \in H \}$. This agrees w. Thm.

Example 1: Let's use (1) to decompose $V = \text{Ind}_H^G \text{triv}$ w. $H = \{1, h\}$ into the direct sum of irreducibles (that also can be done using the techniques of Sec 2.2 in Lec 14). Here we assume char $F = 0$ & $F$ is algebraically closed. Since char $F = 0$, for general $(H, U)$ the r.h.s. in (1) can be rewritten as

$$\frac{1}{|H|} \sum_{k \in G, k^{-1}g \in H} X_U(k^{-1}g) (\text{note that } g_i^{-1} g_i \in H \Leftrightarrow k^{-1} g_k \in H \forall k \in g_i H)$$

Back to $H = \{1, h\}$ & $U = \text{triv}$, note: $X_U(g) = 0$ unless $g = e$, in which case $X_U(e) = \dim V = \frac{1}{2} |G|$, or $g$ is conjugate (in $G$) to $h$, where $X_U(g) = X_U(h) = \frac{1}{2} \{k \in G \mid k^{-1} h k = h \} = \frac{1}{2} |Z_G(h)|$.

Let $W$ be an irreducible representation of $G$. Recall (see Application 2 in Sec 1 of Lec 11) that the multiplicity of
\[ W \text{ in } V \text{ equals } (x_w, x_v) = \frac{1}{|G|} \sum_{g \in G} x_w(g) x_v(g^{-1}). \]

Let \( C = \{ g h g^{-1} | g \in G \} \) so that \( |C| = |G|/|Z_G(h)|. \)

Then \( (x_w, x_v) = \frac{1}{|G|} (x_w(e) \frac{|G|}{2} + |C| x_w(h) \frac{1}{2} |Z_G(h)|) \)

\[ = \frac{1}{2} (\dim W + x_w(h)). \]

Now apply this to \( G = S_4 \) & \( H = S_2 = \{e, (12)\} \). We use the character table for \( S_4 \) from Sec 2.2 of Lec 8 to conclude

\[ \text{Ind}_{S_2}^{S_4} \text{ triv} \cong \text{triv} \oplus (E^4)^{\epsilon_1} \oplus V_2 \oplus \text{sgn} \oplus E^4. \]

And here's an application of Theorem.

**Lemma (tensor identity):** Suppose \( F \) is algebraically closed field of char 0. Let \( U \) & \( V \) be finite dimensional representations of \( H \) & \( G \), respectively. We have

\[ V \otimes \text{Ind}_H^G U \cong \text{Ind}_H^G(V \otimes U), \]

an isomorphism of representations of \( G \).

**Proof:** Recall, Application 1 in Sec 1 of Lec 11, that two
representations are isomorphic if their characters are equal.

\[ X_{V \otimes \text{Ind}_H^G} u(g) = \left[ \text{Sec 1.5 or Addendum of Lec 11} \right] = X_V(g) X_{\text{Ind}_H^G} u(g) \]

\[ X_{\text{Ind}_H^G}(V \circ u)(g) = \sum_{g_i g_j \in H} X_{V \circ u}(g_i g_j g_i) = \sum X_V(g_i g_j g_i) X_u(g_i g_j g_i) \]

\[ = [X_V(g_i g_j) = X_V(g)] X_u(g_i g_j) = X_V(g) X_{\text{Ind}_H^G} u(g) \]

**Remark 1:** The conclusion holds w/o restriction on \( F \) but the argument needs a few things from category theory. It’s based on the following natural isomorphisms:

\[ \text{Hom}_c(V', \text{Ind}_H^G(V \circ u)) \sim \left[ \text{Frobenius reciprocity} \right] \text{Hom}_H(V', V \circ u) \sim \left[ \text{tensor-Hom adjunction} \right] \text{Hom}_H(V \circ V^*, U) \sim \left[ \text{Frobenius reciprocity} \right] \text{Hom}_c(V \circ V^* \circ \text{Ind}_H^G u) \sim \left[ \text{tensor-Hom adjunction} \right] \text{Hom}_c(V', V \otimes \text{Ind}_H^G u) \]

**Remark 2:** Theorem has a fun application to the structure theory of finite groups. Let \( G \) be a finite group. By a Frobenius complement we mean a subgroup \( H \subset G \) which is “as far from being normal as possible”: \( H \cap gHg^{-1} = \{e\} \not\forall g \in G \setminus H \)
Consider the subset \( K := \left( \{ \gamma \} \cup \{ g H g^{-1} \} \right) \cup \{ e \} \). Then it is a subset (automatically normal) – this was proved by Frobenius.

2.3) Curious observations about \( \text{Ind}_{S_4} S_4 \text{triv}, \text{Ind}_{S_4} S_4 \text{sgn} \)

Here’s a complete list of decompositions of \( \text{Ind}_{S_4} S_4 \text{triv} \) into irreducibles. As before, \( \mathbb{F} \) is algebraically closed & \( \text{char} \mathbb{F} = 0 \).

- \( \lambda = (4) \): \( \text{Ind}_{S_4} S_4 \text{triv} = \text{triv} \)

- \( \lambda = (3,1) \): \( \text{Ind}_{S_3} S_4 \text{triv} = \text{Fun} \left( S_4 / S_3, \mathbb{F} \right) = \left[ S_4 / S_3 \rightarrow \{1,2,3,4\}, \right. \)

\[ \text{exercise} = \mathbb{F}^4 = \left[ \text{Lemma in Sec 1.2 of Lec 5} \right] = \text{triv} \oplus \mathbb{F}^4 \]

\[ \text{permutation rep} \]

- \( \lambda = (2,2) \): \( \text{Ind}_{S_4} S_4 \text{triv} = \text{triv} \oplus \mathbb{F}^4 \oplus V_2 \), Sec 2.1 of Lec 14.

- \( \lambda = (2,1,1) \): \( \text{Ind}_{S_4} S_4 \text{triv} = \text{triv} \oplus (\mathbb{F}^4) \otimes V_2 \oplus \text{sgn} \oplus \mathbb{F}^4 \), Example 1.

- \( \lambda = (1,1,1,1) \): \( \text{Ind}_{e_3} S_4 \text{triv} = \text{Fun} (\mathbb{C}, \mathbb{F}) = \left[ \text{regular representation, see Thm in Sec 2.1 of Lec 7} \right] = \text{triv} \oplus (\mathbb{F}^4) \otimes V_2 \oplus (\text{sgn} \oplus \mathbb{F}^4) \otimes \text{sgn} \).

We record this as a table, where rows correspond to \( \text{Ind}_{S_4} S_4 \text{triv} \), columns to the irreducibles & the entries are the multiplicities (we skip 0’s).
### Table 1: Decompositions of $\text{Ind}_{S_4}^{S_4} \text{triv}$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$V$</th>
<th>triv</th>
<th>$F_0^4$</th>
<th>$V_2$</th>
<th>$\text{sgn} \otimes F_0^4$</th>
<th>$\text{sgn}$</th>
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<td>4</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(3,1)$</td>
<td></td>
<td>1</td>
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<td>1</td>
<td></td>
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<tr>
<td>$(2,2)$</td>
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<td>1</td>
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<td>1</td>
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<tr>
<td>$(2,1,1)$</td>
<td></td>
<td>1</td>
<td></td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$(1,1,1,1)$</td>
<td></td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Now we proceed to $\text{Ind}_{S_4}^{S_4} \text{sgn}$, where by the sign representation of $S_4$ we mean the restriction of $\text{sgn}$ from $S_4$. By Lemma in Sec 2.2 (tensor identity), we have $\text{Ind}_{S_4}^{S_4} \text{sgn} \cong \text{sgn} \otimes \text{Ind}_{S_4}^{S_4} \text{triv}$.

So we get the following table.

### Table 2: Decompositions of $\text{Ind}_{S_4}^{S_4} \text{sgn}$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$V$</th>
<th>triv</th>
<th>$F_0^4$</th>
<th>$V_2$</th>
<th>$\text{sgn} \otimes F_0^4$</th>
<th>$\text{sgn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,1,1,1)$</td>
<td></td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
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<tr>
<td>$(2,1,1)$</td>
<td></td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
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<tr>
<td>$(2,2)$</td>
<td></td>
<td>1</td>
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<tr>
<td>$(3,1)$</td>
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<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$(4)$</td>
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