Lecture 15, Induced representations

0) Recap & plan

1) Proof of Frobenius reciprocity.
2) Character formula for the induced representation.

Ref: Secs 5.8-5.10 in [E]
Section 2.1 significantly modified on 3/9

0) Recap & plan

Let $F$ be a field, $H < G$ be finite groups and $U$ be a (finite dimensional) representation of $H$. In Sec 1.1 of Lec 14 we have constructed the induced representation of $G$:

$$\text{Ind}_H^G U = \text{Map}_H (G, U) = \{ \phi : G \to U \mid \phi(gh^{-1}) = h_u \phi(g) \}$$

with $[g, \phi](g) = \phi(g^{-1}g)$.

In Sec 2.1 of Lec 14, we have stated the important property, the Frobenius reciprocity:

for (finite dimensional) representations $U$ of $H$ & $V$ of $G$ we have a natural vector space isomorphism

$$\text{Hom}_G (V, \text{Ind}_H^G U) \cong \text{Hom}_H (V, U)$$  \[ (*) \]
The plan for this lecture is as follows:

- Prove Frobenius reciprocity.
- State & prove the Frobenius formula for the characters of induced representations.
- Make curious observations about representations of the form $\text{Ind}_{S_n}^{S_k}$ for $n=4$ that will be generalized to arbitrary $n$ after the break leading to the classification of irreducible representations of $S_n$.

1) Proof of Frobenius reciprocity.

- First, we construct a map in $(*).$ Define the map

$$\text{ev} : \text{Ind}_H^G U = \text{Map}_H (\xi, U) \to U, \ \text{ev}(\varphi) : = \varphi(e)$$

We claim that it is a homomorphism of representations of $H$:

$$\text{ev} (h. \varphi) = [h. \varphi](e) = \varphi(h^{-1}) = [\varphi \text{ is equivariant}] = h. \varphi(e).$$

Now our map $(*)$ is $\varphi \mapsto \text{ev} \circ \varphi.$ Since $\text{ev}$ & $\varphi$ are homomorphisms of representations of $H,$ so is their composition. Hence we indeed get a well-defined linear map in $(*)$.

- Now we are going to construct an inverse. Let $\varphi \in \text{Ind}_H^G U$ and $\varphi CD$
\[ \text{Hom}_H(V, U). \text{ We are going to define } \psi : V \rightarrow \text{Map}_H(G, U). \text{ We do this by } \psi_2(v)(g) = \varphi(g^{-1}v). \text{ We need to check that } \]

- \[ \psi_2(v) \in \text{Map}_H(G, U) \iff [\psi_2(v)](gh^{-1}) = h_u([\psi_2(v)](g)) \]
- \[ [\psi_2(v)](gh^{-1}) = \varphi(gh^{-1}, v) = h_u \varphi(g^{-1}v) = h_u([\psi_2(v)](g)) \text{ for all } v \in V, \]
- \[ \psi_2 \text{ is } G \text{-equivariant: } \psi_2(g^{-1}v) = g \map_{\psi_2}(g) \psi_2(v), \forall g \in G. \]

\[ [g \map_{\psi_2}(e)](g) = \varphi(g^{-1}v, g) \text{ - checked.} \]

- We need to show that \( \varphi \mapsto \psi_2 \) & \( \psi \mapsto \text{ev} \circ \psi \) are inverse to each other: \( \text{ev} \circ \psi_2 = \varphi \) & \( \psi \circ \text{ev} \circ \psi = \psi \)

\[ \text{ev} \circ \psi_2(v) = [\psi_2(v)](e) = \varphi(v) \text{ for all } v. \]

\[ [\psi \circ \text{ev} \circ \psi(v)](g) = \text{ev}(\psi(g^{-1}v)) = [\psi(g^{-1}v)](e) = [\psi \text{ is } G \text{-equivariant}] \]

\[ = [\psi(v)](g) \Rightarrow \psi \circ \text{ev} \circ \psi = \psi. \]

\[ \square \]

2) Character formula for the induced representation.

2.1) Main result.

Let \( g_i, i=1,...,l \), be representatives of the cosets in \( G/H \), so that \( l = |G/H| \). Every element of \( G \) is uniquely written as \( g_i h^{-1}, i=1,...,l, h \in H. \)
Theorem (Frobenius)

\[ X_{\text{Ind}_H^G}(g) = \sum_{i \mid g_i \cdot g g_i \in H} X_U(g_i \cdot g g_i) \]  

Proof:

In the proof we'll use the following observation. Let \( V \) be a finite dimensional vector space with direct sum decomposition \( V = U_1 \oplus \cdots \oplus U_l \). Let \( x \in \text{End}(V) \). For \( i, j = 1, \ldots, l \), set \( d_{ij} = \gamma_i \circ \pi_j \), where \( \gamma_i : V \to U_i \) is the projection. Then

\[ \text{tr} \, x = \sum_{i=1}^l \text{tr} \, d_{ii}. \]  

We apply this to: \( V = \text{Map}_H(G, U) \), \( \gamma : = g \), and the spaces \( U_i := \{ \phi \in \text{Map}_H(G, U) \mid \phi(g) \neq 0 \Rightarrow \phi \in g_i \cdot H \} \), \( i = 1, \ldots, l \). Note that \( \gamma_i (\phi) \) is the map that coincides with \( \phi \) on \( g_i \cdot H \) and is 0 on the other cosets.

Recall that for \( g \in G \), we have \([g \cdot \phi](g_i \cdot h^{-1}) = \phi(g_i^{-1}g_i h^{-1})\). Suppose \( \phi \in U_i \). If \( g_i^{-1}g_i \notin g_i \cdot H \), then \( \phi(g_i^{-1}g_i h^{-1}) = 0 \), so \( \gamma_i (g_i \cdot \phi) = 0 \). The condition \( g_i^{-1}g_i \in g_i \cdot H \) is equivalent to \( g_i^{-1} g_i g_i \in H \). From (†) we conclude \( X_U(g) = \text{tr} \, x = \sum_{i \mid g_i \cdot g g_i \in H} \text{tr} \, d_{ii} \).

So, we need to prove that if \( g_i^{-1} g g_i \in H \), then \( \text{tr} \, d_{ii} = X_U(g_i^{-1} g g_i) \).
Note that the map \( \varphi \mapsto \varphi(g_i) \) identifies \( U_i \) w. \( U \) (compare to Sec 1.2 of Lec 14). We claim that, under this identification, \( d_{ii} \) coincides with \((g_i^{-1}gg_i)_U \Leftrightarrow [d_{ii}(\varphi)](g_i) = (g_i^{-1}gg_i)_U \varphi(g_i) \) (2)

this will finish the proof: if \( \iota : U \rightarrow U' \) is an isomorphism of finite dimensional vector spaces & \( \varphi \in \text{End}(U), \lambda \in \text{End}(U') \) satisfy \( \lambda \circ \varphi = \varphi \circ \iota \), then \( \text{tr}(\lambda) = \text{tr}(\lambda \circ \varphi \circ \iota^{-1}) = \text{tr}(\varphi) \).

Let \( \varphi \in U_i \). Then the image of \( d_{ii}(\varphi) \) in \( U \) is

\[
[d_{ii}(\varphi)](g_i) = [d_{ii}(\varphi)]_{g_i H} = g_i \varphi |_{g_i H} = [g_i \varphi](g_i) = \varphi(g_i^{-1}g_i) = \varphi(g_i^{-1}g_i) = [g_i^{-1}g_i \in H \& \varphi(g_i h^{-1}) = h_u \varphi(g_i)] = (g_i^{-1}g_i)_U \varphi(g_i), \text{ which gives (2) and finishes the proof} \]

Remark: suppose that \( |H| \) is invertible in \( F \). Then we can rewrite (1) as

\[
X_{\text{Ind}_H^U}(g) = \frac{1}{|H|} \sum_{k \in G} X_U(k^{-1}g_k) \] (1')

This is because for \( k = g_i h^{-1} \), we have \( k^{-1}g_k = h(g_i^{-1}gg_i)^{-1} \), so

\[
X(k^{-1}g_k) = X(g_i^{-1}gg_i) \text{ and therefore the sum in (1') is } |H| \text{ times the sum in (1). Sometimes, (1') is more convenient b/c it does not involve artificial choices.}
2.2) Examples & applications.

Example 1: Suppose $U$ is the 1-dimensional trivial representation. As we have seen in Sec 1.1 of Lec 14,

$$\text{Ind}_H^G U \cong \text{Fun}(G/H, F)$$

By Sec 2.1 of Lec 8, $\chi_{\text{Fun}(G/H, F)}(g) = |\{x \in G/H \mid g \cdot x = x^g\}| = |\{i \in \mathbb{F} \mid g^i \cdot g_i \in g_i H \Leftrightarrow g_i^{-1} \cdot g_i \in H \}|$. This agrees w/ Thm.

Example 1': Let's use (1) to decompose $V = \text{Ind}_H^G \text{triv} w. H = \{1, h\}$ into the direct sum of irreducibles (that also can be done using the techniques of Sec 2.2 in Lec 19). Here we assume $\text{char } F = 0$ & $F$ is algebraically closed. In particular, we can use (1').

Back to $H = \{1, h\}$ & $U = \text{triv}$, note: $X_v(g) = 0$ unless $g$ is conjugate to 1 or $h$. In the 1st case $g = e \Rightarrow X_v(e) = \dim V = |G/H| = \frac{1}{2} |G|$. In the 2nd case,

$$X_v(g) = X_v(h) = \frac{1}{2} |\{\kappa \in G \mid \kappa^{-1} h \kappa = h \}| = \frac{1}{2} |X_v(h)|.$$

Let $W$ be an irreducible representation of $G$. Recall (see Application 2 in Sec 1 of Lec 11) that the multiplicity of
\[ W \text{ in } V \text{ equals } (x_w, x_v) = \frac{1}{|G|} \sum_{g \in G} x_w(g) x_v(g^{-1}). \]

Let \( C = \{ ghg^{-1} | g \in G \} \) so that \( |C| = |G|/|Z_G(h)| \).
Then \( (x_w, x_v) = \prod_{h=h^{-1}} = \frac{1}{|G|} (x_w(e) \frac{|G|}{2} + |C| x_w(h) \frac{1}{2} |Z_G(h)|) = \)
\[ = \frac{1}{2} (\dim W + x_w(h)). \]

Now apply this to \( G = S_4 \) & \( H = S_2 = \{ e, (12) \} \). We use the character table for \( S_4 \) from Sec. 2.2 of Lec 8 to conclude
\[ \text{Ind}_{S_2}^{S_4} \text{ triv} \simeq \text{triv} \oplus (F_4)^e \oplus V_2 \oplus \text{sgn} \otimes F_4. \]

And here's an application of Theorem.

**Lemma (tensor identity):** Suppose \( F \) is algebraically closed field of char 0. Let \( U \) & \( V \) be finite dimensional representations of \( H \) & \( C \), respectively. We have
\[ V \otimes \text{Ind}_H^G U \simeq \text{Res}_H^G (V \otimes U), \]
an isomorphism of representations of \( C \).

**Proof:** Recall, Application 1 in Sec 1 of Lec 11, that two
representations are isomorphic if their characters are equal.

\[ X_{V \otimes \text{Ind}^c_H U}(g) = [\text{Sec 1.5 or Addendum of Lec 11}] = X_V(g) X_{\text{Ind}^c_H U}(g) \]
\[ X_{\text{Ind}^c_H (V \otimes U)}(g) = \sum_{i, g_i g_i \in H} X_{V \otimes U}(g g_i) = \sum X_V(g g_i) X_U(g g_i) \]
\[ = [X_V(g g_i) = X_V(g)] = X_V(g) \sum X_U(g g_i) = X_V(g) X_{\text{Ind}^c_H U}(g) \]

Remark 1: The conclusion holds w/o restriction on \( H \) but the argument needs a few things from category theory. It's based on the following natural isomorphisms:

\[ \text{Hom}_c(V, \text{Ind}^c_H (V \otimes U)) \sim \text{[Frobenius reciprocity]} \text{ Hom}_H (V, V \otimes U) \sim \text{[tensor-Hom adjunction]} \text{ Hom}_H (V \otimes V^*, U) \sim \text{[Frobenius reciprocity]} \text{ Hom}_c (V \otimes V^*, \text{Ind}^c_H U) \sim \text{[tensor-Hom adjunction]} \text{ Hom}_c (V, V \otimes \text{Ind}^c_H U) \]

Remark 2: Theorem has a fun application to the structure theory of finite groups. Let \( G \) be a finite group. By a Frobenius complement we mean a subgroup \( H \subset G \) which is "as far from being normal as possible": \( H \cap gHg^{-1} = \{e\} \forall g \in G \setminus H \).
Consider the subset $K = \left( \{ \lambda G \} \cup \{ \lambda G^2 \} \right) \cup \{e\}$. Then it is a subgroup (automatically normal) – this was proved by Frobenius.

2.3) Curious observations about $Ind_{S_4}^{S_6}$ triv, $Ind_{S_4}^{S_6}$ sgn.

Here’s a complete list of decompositions of $Ind_{S_4}^{S_6}$ triv into irreducibles. As before, $\mathbb{F}$ is algebraically closed \& char $\mathbb{F} = 0$.

- $\lambda = (4)$: $Ind_{S_4}^{S_6}$ triv = triv
- $\lambda = (3,1)$: $Ind_{S_3}^{S_4}$ triv = Fun $(S_4/S_3, \mathbb{F}) = [S_4/S_3 \mapsto \{1,2,3,4\}, \text{exercise}] = \mathbb{F}^d = [\text{Lemma in Sec 1.2 of Lec 5}] = \text{triv} \oplus \mathbb{F}_6^4$
- $\lambda = (2,2)$: $Ind_{S_4}^{S_6}$ triv = triv $\oplus \mathbb{F}_6^2 \oplus V_\chi$, Sec 2.1 of Lec 14.
- $\lambda = (2,1,1)$: $Ind_{S_4}^{S_6}$ triv = triv $\oplus (\mathbb{F}_6^4)^{\otimes 2} \oplus V_\chi \oplus \text{sgn} \oplus \mathbb{F}_6^4$, Example 1!
- $\lambda = (1,1,1,1)$: $Ind_{\{e\}}^{S_4}$ triv = Fun $(C_{12}, \mathbb{F})$ = [regular representation, see Thm in Sec 2.1 of Lec 7] = triv $\oplus (\mathbb{F}_6^4)^{\otimes 3} \oplus V_\chi^{\otimes 5} \oplus (\text{sgn} \oplus \mathbb{F}_6^4)^{\otimes 3} \oplus \text{sgn}$.

We record this as a table, where rows correspond to $Ind_{S_4}^{S_6}$ triv, columns to the irreducibles \& the entries are the multiplicities (we skip 0’s).
Table 1: Decompositions of $\text{Ind}_{S_4}^{S_4} \text{triv}$:

<table>
<thead>
<tr>
<th>$\lambda \backslash V$</th>
<th>triv</th>
<th>$F_0^4$</th>
<th>$V_2$</th>
<th>$\text{sgn} \otimes F_0^4$</th>
<th>$\text{sgn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3,1)</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,2)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
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<tr>
<td>(2,1,1)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(1,1,1,1)</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Now we proceed to $\text{Ind}_{S_4}^{S_4} \text{sgn}$, where by the sign representation of $S_4$ we mean the restriction of $\text{sgn}$ from $S_4$. By Lemma in Sec 2.2 (tensor identity), we have $\text{Ind}_{S_4}^{S_4} \text{sgn} \cong \text{sgn} \otimes \text{Ind}_{S_4}^{S_4} \text{triv}$. So we get the following table.

Table 2: Decompositions of $\text{Ind}_{S_4}^{S_4} \text{sgn}$:

<table>
<thead>
<tr>
<th>$\lambda \backslash V$</th>
<th>triv</th>
<th>$F_0^4$</th>
<th>$V_2$</th>
<th>$\text{sgn} \otimes F_0^4$</th>
<th>$\text{sgn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1,1,1)</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
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<tr>
<td>(2,1,1)</td>
<td>1</td>
<td>1</td>
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<td>2</td>
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<td>(2,2)</td>
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<tr>
<td>(3,1)</td>
<td>1</td>
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<td>(4)</td>
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