Lecture 15, Induced representations.

0) Recap & plan

1) Proof of Frobenius reciprocity.

2) Character formula for the induced representation.

Ref: Secs 5.8-5.10 in [E]

Section 2.1 significantly modified on 3/9.

0) Recap & plan

Let $\mathbb{F}$ be a field, $H \subset G$ be finite groups and U be a (finite dimensional) representation of H. In Sec 1.1 of Lec 14 we have constructed the induced representation of G:

$$\mathrm{Ind}_H^G U \left(= \mathrm{Map}_H(G,U)\right) = \{\varphi: G \to U \mid \varphi(gh^{-1}) = h_U \varphi(g)\}$$

w. $[g_1 \cdot \varphi](g) = \varphi(g_1^{-1} g)$.

In Sec 2.1 of Lec 14, we have stated the important property, the Frobenius reciprocity:

for (finite dimensional) representations U of H & V of G we have a natural vector space isomorphism

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U) \xrightarrow{\sim} \mathrm{Hom}_H(V, \underbrace{U}_{=\mathrm{Res}_H^G U}) \qquad (*)$$

The plan for this lecture is as follows:

- Prove Frobenius reciprocity.
- State & prove the Frobenius formula for the characters of induced representations.
- Make curious observations about representations of the form $\mathrm{Ind}_{S_\lambda}^{S_n} \mathrm{triv}$ & $\mathrm{Ind}_{S_\lambda}^{S_n} \mathrm{sgn}$ for $n=4$ that will be generalized to arbitrary n after the break leading to the classification of irreducible representations of S_n.

1) Proof of Frobenius reciprocity.

- First, we construct a map in (*). Define the map

$$ev: \mathrm{Ind}_H^G U = \mathrm{Map}_H(G,U) \longrightarrow U,\ ev(\varphi) := \varphi(e)$$

We claim that it is a homomorphism of representations of H:

$ev(h.\varphi) = [h.\varphi](e) = \varphi(h^{-1}) = [\varphi \text{ is equivariant}] = h_U.\varphi(e)$.

Now our map (*) is $\psi \mapsto ev \circ \psi$. Since ev & ψ are homomorphisms of representations of H, so is their composition. Hence we indeed get a well-defined linear map in (*).

- Now we are going to construct an inverse. Let $\eta \in$

$Hom_H(V,U)$. We are going to define $\psi_\eta : V \to Map_H(G,U)$. We do this by $[\psi_\eta(v)](g) := \eta(g_V^{-1}v)$. We need to check that

- $\psi_\eta(v) \in Map_H(G,U) \Leftrightarrow [\psi_\eta(v)](gh^{-1}) = h_U([\psi_\eta(v)](g))$.

$[\psi_\eta(v)](gh^{-1}) = \eta(h_V g_V^{-1} v) = h_U\, \eta(g_V^{-1} v) = h_U([\psi_\eta(v)](g))$ ✓

- ψ_η is G-equivariant: $\psi_\eta(g'_V v) = g'_{Map_H(G,U)} \psi_\eta(v)$, $\forall g' \in G$.

$\psi_\eta(g'_V v)(g) = \eta(g_V^{-1} g'_V v)$

$[g'_{Map_H(G,U)} \psi_\eta(v)](g) = [\psi_\eta(v)](g'^{-1}g) = \eta(g_V^{-1} g'_V v)$ - checked.

- We need to show that $\eta \mapsto \psi_\eta$ & $\psi \mapsto ev\circ\psi$ are inverse to each other: $ev\circ\psi_\eta = \eta$ & $\psi_{ev\circ\psi} = \psi$.

$ev\circ\psi_\eta(v) = [\psi_\eta(v)](e) = \eta(v)$ ✓.

$[\psi_{ev\circ\psi}(v)](g) = ev(\psi(g_V^{-1}v)) = [\psi(g_V^{-1}v)](e) = [\psi \text{ is } G\text{-equiv't}]$

$= [\psi(v)](g) \Rightarrow \psi_{ev\circ\psi} = \psi$ $\square$

2) Character formula for the induced representation.

2.1) Main result.

Let g_i, $i=1,\dots,\ell$, be representatives of the cosets in G/H, so that $\ell = |G/H|$. Every element of G is uniquely written as $g_i h^{-1}$, $i=1,\dots\ell$, $h\in H$.

Theorem (Frobenius)

$$\chi_{\mathrm{Ind}_H^G U}(g) = \sum_{i \mid g_i^{-1} g g_i \in H} \chi_U(g_i^{-1} g g_i) \qquad (1)$$

Proof:

In the proof we'll use the following observation. Let V be a finite dimensional vector space w. direct sum decomposition $V = U_1 \oplus \ldots \oplus U_\ell$. Let $\alpha \in \mathrm{End}(V)$. For $i,j = 1,\ldots,\ell$, set $\alpha_{ij} := \pi_i \circ \alpha|_{U_j}$, where $\pi_i : V \twoheadrightarrow U_i$ is the projection. Then

$$\mathrm{tr}\,\alpha = \sum_{i=1}^{\ell} \mathrm{tr}\,\alpha_{ii} \qquad (\dagger)$$

We apply this to: $V = \mathrm{Map}_H(G,U)$, $\alpha := g_V$, and the spaces $U_i := \{\varphi \in \mathrm{Map}_H(G,U) \mid \varphi(g) \neq 0 \Rightarrow g \in g_i H\}$, $i = 1,\ldots,\ell$. Note that $\pi_i(\varphi)$ is the map that coincides w. φ on $g_i H$ and is 0 on the other cosets.

Recall that for $g \in G$, we have $[g.\varphi](g_i h^{-1}) = \varphi(g^{-1} g_i h^{-1})$. Suppose $\varphi \in U_i$. If $g^{-1} g_i \notin g_i H$, then $\varphi(g^{-1} g_i h^{-1}) = 0$, so $\pi_i(g.\varphi) = 0$. The condition $g^{-1} g_i \in g_i H$ is equivalent to $g_i^{-1} g^{-1} g_i \in H \Leftrightarrow g_i^{-1} g g_i \in H$. From $(\dagger)$ we conclude $\chi_V(g) = \mathrm{tr}\,\alpha = \sum_{i \mid g_i^{-1} g g_i \in H} \mathrm{tr}\,\alpha_{ii}$

So, we need to prove that if $g_i^{-1} g g_i \in H$, then $\mathrm{tr}\,\alpha_{ii} = \chi_U(g_i^{-1} g g_i)$.

Note that the map $\varphi \mapsto \varphi(g_i)$ identifies U_i w. U (compare to Sec 1.2 of Lec 14). We claim that, under this identification, α_{ii} coincides with $(g_i^{-1} g g_i)_U \Leftrightarrow [\alpha_{ii}(\varphi)](g_i) = (g_i^{-1} g g_i)_U\, \varphi(g_i)$ (2)
this will finish the proof: if $\iota: U \xrightarrow{\sim} U'$ is an isomorphism of finite dimensional vector spaces & $\beta \in End(U)$, $\alpha \in End(U')$ satisfy $\iota\circ\beta = \alpha\circ\iota$, then $tr(\alpha) = tr(\iota\circ\beta\circ\iota^{-1}) = tr(\beta)$.

Let $\varphi \in U_i$. Then the image of $\alpha_{ii}(\varphi)$ in U is

$[\alpha_{ii}(\varphi)](g_i) = [\alpha_{ii}(\varphi)|_{g_iH} = g.\varphi|_{g_iH}] = [g.\varphi](g_i) = \varphi(g^{-1}g_i) =$
$= \varphi(g_i (g_i^{-1} g g_i)^{-1}) = [g_i^{-1} g g_i \in H \ \&\ \varphi(g_i h^{-1}) = h_U . \varphi(g_i)] =$
$= (g_i^{-1} g g_i)_U . \varphi(g_i)$, which gives (2) and finishes the proof $\square$

Remark: suppose that $|H|$ is invertible in $\mathbb{F}$. Then we can rewrite (1) as

$$\chi_{Ind_H^G U}(g) = \frac{1}{|H|} \sum_{k\in G | k^{-1}gk \in H} \chi_U(k^{-1}gk) \qquad (1')$$

This is because for $k = g_i h^{-1}$, we have $k^{-1}gk = h(g_i^{-1} g g_i)h^{-1}$, so $\chi(k^{-1}gk) = \chi(g_i^{-1} g g_i)$ and therefore the sum in (1') is $|H|$ times the sum in (1). Sometimes, (1') is more convenient b/c it does not involve artificial choices.

2.2) Examples & applications.

Example 1: Suppose U is the 1-dimensional trivial representation. As we have seen in Sec 1.1 of Lec 14,

$$\mathrm{Ind}_H^G U \xrightarrow{\sim} \mathrm{Fun}(G/H,\mathbb{F})$$

By Sec 2.1 of Lec 8, $\chi_{\mathrm{Fun}(G/H,\mathbb{F})}(g) = |\{x\in G/H \mid g.x=x\}| = |\{i\in\{1,\dots,\ell\} \mid g^{-1}g_i\in g_iH \Leftrightarrow g_i^{-1}gg_i\in H\}|$. This agrees w. Thm.

Example 1': Let's use (1) to decompose $V:=\mathrm{Ind}_H^G \mathrm{triv}$ w. $H=\{1,h\}$ so $h=h^{-1}$ into the direct sum of irreducibles (that also can be done using the techniques of Sec 2.2 in Lec 14). Here we assume $\mathrm{char}\,\mathbb{F}=0$ & $\mathbb{F}$ is algebraically closed. In particular, we can use (1').

Back to $H=\{1,h\}$ & $U=\mathrm{triv}$, note: $\chi_V(g)=0$ unless g is conjugate to 1 or h. In the 1st case $g=e \Rightarrow \chi_V(e)=\dim V = |G/H| = \frac{1}{2}|G|$. In the 2nd case,

$$\chi_V(g)=\chi_V(h) = \frac{1}{2}|\{k\in G \mid k^{-1}hk=h\}| = \frac{1}{2}|Z_G(h)|.$$

Let W be an irreducible representation of G. Recall (see Application 2 in Sec 1 of Lec 11) that the multiplicity of

W in V equals $(\chi_W, \chi_V) = \frac{1}{|G|}\sum_{g\in G} \chi_W(g)\chi_V(g^{-1})$.

Let $C = \{ghg^{-1} | g \in G\}$ so that $|C| = |G|/|Z_G(h)|$.

Then $(\chi_W, \chi_V) = [h = h^{-1}] = \frac{1}{|G|}\Big(\underbrace{\chi_W(e)}_{\dim W}\frac{|G|}{2} + \underbrace{|C|\chi_W(h)\frac{1}{2}|Z_G(h)|}_{\text{product is } |G|}\Big) =$

$= \frac{1}{2}(\dim W + \chi_W(h))$.

Now apply this to $G = S_4$ & $H = S_2 = \{e, (12)\}$. We use the character table for S_4 from Sec 2.2 of Lec 8 to conclude

$$\mathrm{Ind}_{S_2}^{S_4} \mathrm{triv} \simeq \mathrm{triv} \oplus (\mathbb{F}_0^4)^{\oplus 2} \oplus V_2 \oplus \mathrm{sgn} \otimes \mathbb{F}_0^4.$$

And here's an application of Theorem.

Lemma (tensor identity): Suppose $\mathbb{F}$ is algebraically closed field of char 0. Let U & V be finite dimensional representations of H & G, respectively. We have

$$V \otimes \mathrm{Ind}_H^G U \xrightarrow{\sim} \mathrm{Ind}_H^G(\underbrace{V}_{\mathrm{Res}_H^G V} \otimes U),$$

an isomorphism of representations of G.

Proof: Recall, Application 1 in Sec 1 of Lec 11, that two representations are isomorphic if their characters are equal.

$\chi_{V\otimes \mathrm{Ind}_H^G U}(g) = $ [Sec 1.5 or Addendum of Lec 11] $= \chi_V(g)\,\chi_{\mathrm{Ind}_H^G U}(g)$

$$\chi_{\mathrm{Ind}_H^G (V\otimes U)}(g) = \sum_{i \mid g_i^{-1} g g_i \in H} \chi_{V\otimes U}(g_i^{-1} g g_i) = \sum_{\dots} \chi_V(g_i^{-1} g g_i)\,\chi_U(g_i^{-1} g g_i)$$

$$= [\chi_V(g_i^{-1} g g_i) = \chi_V(g)] = \chi_V(g) \sum_{\dots} \chi_U(g_i^{-1} g g_i) = \chi_V(g)\,\chi_{\mathrm{Ind}_H^G U}(g) \quad \square$$

Remark 1: The conclusion holds w/o restriction on $\mathbb{F}$ but the argument needs a few things from category theory. It's based on the following natural isomorphisms:

$\mathrm{Hom}_G(V', \mathrm{Ind}_H^G(V\otimes U)) \xrightarrow{\sim}$ [Frobenius reciprocity] $\mathrm{Hom}_H(V', V\otimes U)$ $\xrightarrow{\sim}$ [tensor-Hom adjunction] $\mathrm{Hom}_H(V'\otimes V^*, U) \xrightarrow{\sim}$ [Frobenius reciprocity] $\mathrm{Hom}_G(V'\otimes V^*, \mathrm{Ind}_H^G U) \xrightarrow{\sim}$ [tensor-Hom adjunction] $\mathrm{Hom}_G(V', V\otimes \mathrm{Ind}_H^G U)$.

Remark 2: Theorem has a fun application to the structure theory of finite groups. Let G be a finite group. By a Frobenius complement we mean a subgroup $H \subset G$ which is "as far from being normal as possible": $H \cap gHg^{-1} = \{e\}$ $\forall g \in G\setminus H$.

Consider the subset $K := (G \setminus \bigcup_{g\in G} gHg^{-1}) \sqcup \{e\}$. Then it is a subgroup (automatically normal) – this was proved by Frobenius.

2.3) Curious observations about $\mathrm{Ind}_{S_\lambda}^{S_4}$ triv, $\mathrm{Ind}_{S_\lambda}^{S_4}$ sgn.

Here's a complete list of decompositions of $\mathrm{Ind}_{S_\lambda}^{S_4}$ triv into irreducibles. As before, $\mathbb{F}$ is algebraically closed & char $\mathbb{F}=0$.

- $\lambda=(4)$: $\mathrm{Ind}_{S_4}^{S_4} \mathrm{triv} = \mathrm{triv}$
- $\lambda=(3,1)$: $\mathrm{Ind}_{S_3}^{S_4} \mathrm{triv} = \mathrm{Fun}(S_4/S_3, \mathbb{F}) = [S_4/S_3 \xrightarrow{\sim} \{1,2,3,4\}$, exercise$] = \mathbb{F}^4 = [$Lemma in Sec 1.2 of Lec 5$] = \mathrm{triv} \oplus \mathbb{F}_0^4$.

permutation rep

- $\lambda=(2,2)$: $\mathrm{Ind}_{S_\lambda}^{S_4} \mathrm{triv} = \mathrm{triv} \oplus \mathbb{F}_0^4 \oplus V_2$, Sec 2.1 of Lec 14.
- $\lambda=(2,1,1)$: $\mathrm{Ind}_{S_\lambda}^{S_4} \mathrm{triv} = \mathrm{triv} \oplus (\mathbb{F}_0^4)^{\oplus 2} \oplus V_2 \oplus \mathrm{sgn} \otimes \mathbb{F}_0^4$, Example 1.
- $\lambda=(1,1,1,1)$: $\mathrm{Ind}_{\{e\}}^{S_4} = \mathrm{Fun}(G,\mathbb{F}) = [$regular representation, see Thm in Sec 2.1 of Lec 7$] = \mathrm{triv} \oplus (\mathbb{F}_0^4)^{\oplus 3} \oplus V_2^{\oplus 2} \oplus (\mathrm{sgn} \otimes \mathbb{F}_0^4)^{\oplus 3} \oplus \mathrm{sgn}$.

We record this as a table, where rows correspond to $\mathrm{Ind}_{S_\lambda}^{S_4}$ triv, columns to the irreducibles & the entries are the multiplicities (we skip 0's)

Table 1: Decompositions of $\mathrm{Ind}_{S_\lambda}^{S_4}$ triv:

$\lambda \setminus V$	triv	$\mathbb{F}_0^4$	V_2	$sgn \otimes \mathbb{F}_0^4$	sgn
4	1				
(3,1)	1	1			
(2,2)	1	1	1		
(2,1,1)	1	2	1	1	
(1,1,1,1)	1	3	2	3	1

Now we proceed to $\mathrm{Ind}_{S_\lambda}^{S_4} sgn$, where by the sign representation of S_λ we mean the restriction of sgn from S_4. By Lemma in Sec 2.2 (tensor identity), we have $\mathrm{Ind}_{S_\lambda}^{S_4} sgn \simeq sgn \otimes \mathrm{Ind}_{S_\lambda}^{S_4}$ triv. So we get the following table

Table 2: Decompositions of $\mathrm{Ind}_{S_\lambda}^{S_4}$ sgn.

$\lambda \setminus V$	triv	$\mathbb{F}_0^4$	V_2	$sgn \otimes \mathbb{F}_0^4$	sgn
(1,1,1,1)	1	3	2	3	1
(2,1,1)		1	1	2	1
(2,2)			1	1	1
(3,1)				1	1
(4)					1