Lecture 16:

Representations of symmetric groups, I.

1) Main classification result

Ref: [E], Sec. 5.12 & 5.13.

1) Main classification result

Our goals in this part of the course are:

- Classify the irreducible representations of S_n over an algebraically closed field $\mathbb{F}$ w. char $\mathbb{F}=0$ (the restriction on the characteristic is essential, while the condition of being algebraically closed can be dropped)
- Compute the characters of irreducible representations.

There are a number of reasons to care about representations of S_n specifically. Some have to do with the nature of this group:

• $S_n = Bij(\{1,2,\dots n\})$ is one of "distinguished groups," see Lec 1.

• S_n is a "reflection group" and a "Weyl group." See Bonus Lectures A1 & A2.

• S_n contains the alternating group A_n as an index 2 subgroup. It turns out that one can deduce the classification of irreducible representations of A_n from that of S_n - and the latter is nicer. The importance of A_n is that it's a simple group (for $n \geqslant 5$). Simple groups (& groups that are close to simple, such as S_n) play a distinguished role in the structure theory & the represtation theory of finite groups.

The 2nd group of reasons is connections to other objects in Representation theory & beyond. This includes:

• A connection to "polynomial representations" of $GL_m(\mathbb{C})$ via the Schur-Weyl duality - to be reviewed in a future bonus lecture.

• A connection to Combinatorics: Young diagrams/tableaux, symmetric polynomials. We will see some of these connections in the lectures and some more as a bonus.

• Connections to Probability – that we are unlikely to see in any form.

1.1) Combinatorial preparation.

The irreducible representations of S_n are indexed by partitions. In order to state the classification result we'll need some combinatorial constructions. We start with a partial order.

Definition: Let $\lambda = (\lambda_1, \dots \lambda_k)$, $\mu = (\mu_1, \dots \mu_k)$ be partitions of n (we adjoin zero parts to λ & μ to make the number of parts equal). We consider the lexicographic order: $\lambda < \mu$ if $\lambda_1 = \mu_1, \dots \lambda_{i-1} = \mu_{i-1}$ but $\lambda_i < \mu_i$ for some i.

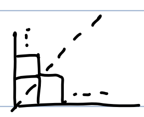
Example: For $n=4$, we have:

$(4) > (3,1) > (2,2) > (2,1,1,1) > (1,1,1,1)$.

Next, we will need an involution on the set of partitions. To define it, we will visualize partitions as Young diagrams: to a partition $(\lambda_1, \dots \lambda_k)$ w. $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k$ we assign the configuration of unit boxes on the plane that are arranged in rows w. λ_1 (1st row), λ_2 (2nd row) etc. boxes aligned on the left:

Example: $(4) \to \ydiagram{4}$, $(3,1) \to \ydiagram{3,1}$, $(2,2) \to \ydiagram{2,2}$, $(2,1,1) \to \ydiagram{2,1,1}$, $(1,1,1,1) \to \ydiagram{1,1,1,1}$.

To a partition, equiv., Young diagram, λ we assign its transpose λ^t: obtained by reflecting the diagram w.r.t. the diagonal:



E.g. $(4)^t = (1,1,1,1)$, $(3,1)^t = (2,1,1)$: $\ydiagram{3,1} \longleftrightarrow \ydiagram{2,1,1}$, $(2,2)^t = (2,2)$.

1.2) Main result.

We are going to use two families of induced representations of S_n to be denoted by I^+_λ, I^-_λ, where λ runs over the set of partitions of n.

Recall that to a partition $\lambda=(\lambda_1,\dots,\lambda_k)$ we assign the subgroup $S_\lambda = S_{\lambda_1}\times S_{\lambda_2}\times\dots\times S_{\lambda_k}\subset S_n$.

Definition: Set $I^+_\lambda := \mathrm{Ind}_{S_\lambda}^{S_n}\mathrm{triv}$, $I^-_\lambda := \mathrm{Ind}_{S_{\lambda^t}}^{S_n}\mathrm{sgn}$.

Theorem: The following claims are true:

1) For every partition λ, $\exists!$ irreducible representation, V_λ, of S_n that occurs in both I^+_λ, I^-_λ.

2) Every irreducible representation of S_n is isomorphic to V_λ for unique λ.

Example: For $n=4$, we have computed the representations $I^\pm_\lambda$ in Sec 2.3 of Lec 15. Here are the results:

$\lambda=(4)$: $I^+_{(4)}=\mathrm{triv}$, $I^-_{(4)}=\mathrm{triv}\oplus(\mathbb{F}_0^4)^{\oplus 3}\oplus V_2^{\oplus 2}\oplus(\mathrm{sgn}\otimes\mathbb{F}_0^4)^{\oplus 3}\oplus\mathrm{sgn}$.

So $V_{(4)}=\mathrm{triv}$.

$\lambda=(3,1)$: $I^+_{(3,1)}=\mathrm{triv}\oplus\mathbb{F}_0^4$, $I^-_{(3,1)}=\mathbb{F}_0^4\oplus V_2\oplus(\mathrm{sgn}\otimes\mathbb{F}_0^4)^{\oplus 2}\oplus\mathrm{sgn}$.

So $V_{(3,1)} = \mathbb{F}_0^4$.

$\lambda=(2,2)$: $I^+_{(2,2)} = triv \oplus \mathbb{F}_0^4 \oplus V_2$, $I^-_{(2,2)} = V_2 \oplus sgn\otimes \mathbb{F}_0^4 \oplus sgn$.

So $V_{(2,2)} = V_2$.

$\lambda=(2,1,1)$: $I^+_{(2,1,1)} = triv \oplus (\mathbb{F}_0^4)^{\oplus 2} \oplus V_2 \oplus sgn\otimes \mathbb{F}_0^4$,

$I^-_{(2,1,1)} = sgn\otimes \mathbb{F}_0^4 \oplus sgn$. So $V_{(2,1,1)} = sgn\otimes \mathbb{F}_0^4$.

$\lambda=(1,1,1,1)$: $I^+_{(1,1,1,1)} = triv \oplus (\mathbb{F}_0^4)^{\oplus 3} \oplus V_2^{\oplus 2} \oplus (sgn\otimes \mathbb{F}_0^4)^{\oplus 3} \oplus sgn$,

$I^-_{(1,1,1,1)} = sgn$. So $V_{(1,1,1,1)} = sgn$.

Exercise: Prove that $sgn\otimes V_\lambda \simeq V_{\lambda^t}$. Hint: use $sgn\otimes I^+_\lambda \simeq I^-_{\lambda^t}$, which follows from Lemma in Sec 2.2 in Lec 15.

1.3) Main technical claim.

We will deduce the classification theorem from the following result

Main Claim: For partitions λ & μ the following are true:

1) $Hom_{S_n}(I^+_\lambda, I^-_\mu) \neq 0 \Rightarrow \mu^t \leq \lambda^t$.

2) $\dim Hom_{S_n}(I^+_\lambda, I^-_\lambda) = 1$.

Example: The explicit computations verify this claim for $n=4$.

This claim will be proved in the next lecture. Now we deduce the theorem from it.

Proof of Theorem modulo the main claim:

Step 1: Let G be an arbitrary finite group and U, V be its finite dimensional representations. Then we can decompose U, V into the direct sums of irreducibles:

$$U = \bigoplus_{i=1}^{\ell} U_i^{\oplus m_i}, \quad V = \bigoplus_{i=1}^{\ell} U_i^{\oplus n_i}.$$

Then (the proof is left as an exercise) we have

$$(*) \quad \dim \operatorname{Hom}_G(U,V) = \sum_{i=1}^{\ell} m_i n_i$$

So (1) of the Main Claim means that if I_λ^+, I_μ^- have common irreducible summands, then $\mu^t \leq \lambda^t$. And (2) means that I_λ^+, I_λ^- share only one common irreducible summand (with multiplicity 1). We denote it by V_λ. This establishes part 1 of the theorem.

Step 2: Now we show that $V_\lambda \simeq V_\mu \Rightarrow \lambda = \mu$. Assume $\lambda \neq \mu$. Then, swapping λ & μ if needed, we can assume $\mu^t > \lambda^t$. Note that V_λ is a direct summand in I^+_λ, while V_μ is a direct summand in I^-_μ. Since $V_\lambda \simeq V_\mu$, we can use (*) above to show that $\mathrm{Hom}_{S_n}(I^+_\lambda, I^-_\mu) \neq 0$. This contradicts (1) of the Main Claim, leading to contradiction.

Step 3: Now we are ready to prove (2) of the theorem. We've got a collection of pairwise non-isomorphic irreducibles indexed by the partitions of n. The same set indexes the conjugacy classes in S_n. As the number of irreducibles (up to isomorphism) equals the number of conjugacy classes (Corollary in Sec 2 of Lec 10), we have actually constructed all irreducible representations, finishing the proof $\square$

1.4) Computation of $\dim \mathrm{Hom}_{S_n}(I^+_\lambda, I^-_\mu)$

The following result is the first step in proving the Main Claim.

Lemma: $\dim \mathrm{Hom}_{S_n}(I^+_\lambda, I^-_\mu)$ coincides w. the dimension of
$\{f\in \mathrm{Fun}(S_n,\mathbb{F})\mid f(\tau g \sigma)=\mathrm{sgn}(\tau)f(g)\ \forall g\in S_n, \tau\in S_{\mu^t}, \sigma\in S_\lambda\}$

Proof: Recall the Frobenius reciprocity: for finite groups $H\subset G$ representations U of H, V of G we have (Sec 2.1 of Lec 14):

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U)\xrightarrow{\sim}\mathrm{Hom}_H(V,U).$$

Apply this to: $G=S_n$, $H=S_{\mu^t}$, $V=I^+_\lambda$, $U=\mathrm{sgn}$. We get

$$\mathrm{Hom}_{S_n}(V, I^-_\mu)\longrightarrow \mathrm{Hom}_{S_{\mu^t}}(V,\mathrm{sgn})$$

The dimension of the target is the multiplicity of sgn in V viewed as a representation of S_{μ^t} that can also be computed as $\dim \mathrm{Hom}_{S_{\mu^t}}(\mathrm{sgn}, V)$, see Sec 1 in Lec 7. The space $\mathrm{Hom}_{S_{\mu^t}}(\mathrm{sgn},V)$ is identified w.

$$\{v\in V\mid \tau v=\mathrm{sgn}(\tau)v\ \forall \tau\in S_{\mu^t}\} \qquad (*)$$

(we send $\varphi:\mathrm{sgn}\to V$ to $\varphi(1)$, the condition that φ is a homomorphism precisely means $\tau\varphi(1)=\mathrm{sgn}(\tau)\varphi(1)$).

Now we use that

$$V=\mathrm{Ind}_{S_\lambda}^{S_n}\mathrm{triv}=\{f\in \mathrm{Fun}(G,\mathbb{F})\mid f(g\sigma)=f(g)\ \forall g\in S_n, \sigma\in S_\lambda\}$$

w. S_n-action given by $[\tau.f](g)=f(\tau^{-1}g)$ (for $\tau\in S_n$). We note that $\operatorname{sgn}(\tau) = \operatorname{sgn}(\tau^{-1})$ and conclude that (*) is:

$\{f \in Fun(G,\mathbb{F}) \mid f(\tau g) = \operatorname{sgn}(\tau) f(g),\ f(g\sigma) = f(g)\ \forall g \in S_n, \sigma \in S_\lambda, \tau \in S_{\mu^t}\}$.

This is exactly the space in the statement of the lemma. □

1.5) Remarks on the classification over other fields.

This discussion is not going to be used in what follows.

Suppose char $\mathbb{F} = 0$ but $\mathbb{F}$ may be non-closed. The proof of Lemma in Sec 1.4 carries over w. slight modifications (left as exercise). In fact, the Main Claim holds over any $\mathbb{F}$ as long as char $\mathbb{F} \neq 2$. Then (*) in Step 1 of the proof in the previous section becomes

$$\dim \operatorname{Hom}_G(U,V) = \sum_{i=1}^{\ell} m_i n_i \dim_{\mathbb{F}} \operatorname{End}_G(U_i)$$

From here we still get the irreducible representation V_λ as in the theorem, and, moreover, $\dim_{\mathbb{F}} \operatorname{End}_{S_n}(V_\lambda) = 1$. Step 2 carries over, so we get a collection of irreducible representations of S_n over $\mathbb{F}$ indexed by partitions. As we remarked in Sec 1.1 of Lec 10, the characters of irreducibles are still orthogonal, hence linearly independent. It follows again that every irreducible representation of $\mathbb{F}S_n$ is isomorphic to V_λ for a unique λ.

Now suppose $\operatorname{char} \mathbb{F} = p$ and $\mathbb{F}$ is algebraically closed (for symmetric groups this assumption is not relevant). There's a general result that for a finite group G, the number of irreducible $\mathbb{F}G$-modules is equal to the number of conjugacy classes whose elements have order coprime to p (we have seen this for p-groups in HW2). For $G = S_n$, one can say more. An element of S_n has order coprime to p iff the lengths of all cycles in its decomposition are coprime to p. And there's a distinguished bijection between the set of irreducible $\mathbb{F}S_n$-modules and the set of partitions of n w/o parts divisible by p.